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UNIT I

Mathematics of Investment

1 Stocks

In addition to their ordinary internal approach to maintaining a solid equity by retaining earnings, business firms, especially large corporations, take an additional external approach to finance their capital needs. To fund their startup operations, maintain their continuous spending, and fund their expansion projects, corporations may raise the needed capital by issuing and selling stocks and bonds. Two major types of securities serve as negotiable instruments of ownership. Investors who are interested in long-term investment buy and sell stocks and bonds in the market and hope to earn money through receiving dividends as their shares of a firm's profits, as well as through making capital gains when their investment values appreciate. To ensure the safety of their investment and take the best path to earning good returns, investors would have to be studious and smart in following the frequent fluctuations of prices and trends in their stocks and bonds. In this chapter we focus on mathematical operations related to stocks, and continue in the following chapters to address bonds and other types of securities.

For corporations, stocks are a major external source of equity capital that have claims on a firm's income and assets secondary to claims by their regular business creditors. For an investor, stocks are shares of ownership in the assets and earnings of a firm, and therefore a source of individual income. There are two major types of stocks; common stocks and preferred stocks. **Common stocks** are the most basic form of business ownership. They represent the great ability that a business has to increase its funding, and at the same time they place a minimum constraint on the firm, and for that, common stockholders are compensated with higher dividends and voting rights, compared with the holders of preferred stock. Voting rights in a corporation means that a stockholder would share in making major policy decisions, election of officers, and evaluation of management and progress. The power of voting would be commensurate with the number of shares that a stockholder holds. Common stocks can be owned privately or publicly and may come in various classes and categories.

Preferred stocks are of the fixed-income type. Stockholders of preferred stocks receive, as a dividend, a fixed percentage or amount of a firm's earnings. The firm is obligated to pay those dividends even before it pays its common stockholders. However, this priority privilege for preferred stocks is countered by the lack of voting rights. Other perceived disadvantages of preferred stocks are their relatively higher costs and the sensitivity of their market price to the fluctuation of interest rate. However, one big advantage of preferred stocks, on the firm's side, is that they tend to increase the firm's leverage, as we will see in detail later.

1.1. BUYING AND SELLING STOCKS

The primary objective for the common stockholders is to earn money, not only by receiving dividends but also by trading and making capital gains through buying and selling the right stock at the right time. The basic premise here is that they would like to buy at a time of **undervalued stocks**. That is when the true value of the stock is expected to be higher than its market value. They would also sell at the time of **overvalued stocks**, meaning that the true value of a stock is considered to be less than what it is sold for in the market.

For that reason, an investor has to be aware not only of the market changes and price fluctuations but also of the commissions and brokerage fees and the way they are calculated. In the stock exchange market, brokerage fees are set based on the market value of stocks and the number of shares purchased or sold. The brokerage rate is usually a combination of fixed and variable costs. The fixed part would increase with the amount of purchase or sale, and the variable part, which is usually a percentage, decreases as the amount of the purchase or sale, goes up, as shown in Table 1.1.

Example 1.1.1 Dale purchased 350 shares of Country Pizza's common stock at \$25.75 per share but later sold 200 shares at \$29.25 each. What would his capital gain and his investment rate of return be?

1. Total cost of investment:

$350 \times \$25.75 = \$9{,}012.50$	initial investment
$\$9{,}012.50 \times .006 = \$54.08$	brokerage % on purchase
$\$70 + \$54.08 = \$124.08$	total brokerage fee
$\$9{,}012.50 + \$124.08 = \$9{,}136.58$	cost of total investment

TABLE 1.1

Amount of	Brokerage Rate		
Purchase or/Sale (\$)	Initial Charge (\$)	+	%
→ 2,500	25		.015
2,501–6,000	50		.007
6,001–22,000	70		.006
22,001–50,000	90		.004
50,001–500,000	150		.002
500,000 →	250		.001

2. Net sale:

$$200 \times \$29.25 = \$5{,}850 \qquad \text{total sale}$$
$$\$5{,}850 \times .007 = \$40.95 \qquad \text{brokerage \% on sale}$$
$$\$50 + \$40.95 = \$90.95 \qquad \text{total brokerage on sale}$$
$$\$5{,}850 - \$90.95 = \$5{,}759.05 \qquad \text{net sale}$$

3. Cost of 200 shares sold:

$$\$9{,}136.50 \times \frac{200}{350} = \$5{,}220.85$$

4. Capital gain:

$$\$5{,}759.05 - \$5{,}220.86 = \$538.19 \qquad \text{capital gain}$$

5. Rate of return:

$$\frac{\$538.19}{\$5{,}220.85} = 10.3\%$$

Note that we did not get the cost of the 200 shares sold initially because that would place the brokerage fee at the second line of the chart. But in reality, he already paid the fees on the entire 350 shares purchased based on the third line of the chart. That was why we just calculated the share of these 200 sold out of what he paid for the total number of shares he purchased (350).

Example 1.1.2 Suppose that a new investor decided to dedicate \$5,000 to purchase a certain stock and pay its fees. If he was told that the commission would be \$50 plus .007, find (1) how many shares he would get if that stock sells for \$28.58, and (2) what would be his yield if that stock pays \$2.33 dividend per share.

First, we have to determine his net investment; that is, we exclude the brokerage fee from his \$5,000.

$$\mathrm{NI} + .007\mathrm{NI} + 50 = \$5{,}000$$
$$\mathrm{NI}(1 + .007) = \$4{,}950$$
$$\mathrm{NI} = \frac{\$4{,}950}{\mathbf{1.007}} = \$4{,}915.60 \qquad \text{net investment}$$

Since the stock price per share is \$28.50, we would find the number of shares:

$$\frac{\$4{,}915.60}{\$28.58} = 172 \text{ shares}$$

His total dividends would be

$$172 \times \$2.33 = \$400.76$$

The yield on his investment is

$$\frac{\$400.76}{\$5,000} = 8\%$$

1.1.1 COMMON STOCK VALUATION

The estimation of stock value is needed whether or not the stock is actively traded in the stock exchange. It is important to find the price or value of securities such as the stocks, not only for buying and selling purposes, but also for other financial purposes such as estate tax valuation and new-share insurance. The main premise in consideration of the current value of a share of common stock is that it should be estimated by the present value of its future cash flows, which are the dividends that would come in the future. If an investor purchases a certain stock for a price of P_0, she would expect to receive dividends D on this investment as long as she keeps holding the shares. Stocks do not have any maturity date, so the future of receiving dividends is open. Another investor's expectation would be the price appreciation (PA) of her stock prices in the market that would bring a capital gain through selling at a higher price. The formal rate of return that investors expect to receive on their investment is called the **market capitalization rate** (MCR), or **expected rate** (Er), or just the **rate of return** (r). It is determined by

$$\text{MCR} = \text{Er} = r = \frac{D + PA}{P_0}$$

where D is the expected dividend per share and PA is the stock price appreciation that would be obtained by getting the difference between the current purchase price P_0 and the expected price (P_1) of that stock a year after the purchase.

$$\text{PA} = P_1 - P_0$$

We can, therefore, adjust the formula to

$$\boxed{r = \frac{D + P_1 - P_0}{P_0}}$$

That is, an investor's expected annual return. It is weighing two typical types of returns, the expected cash dividend per share and the capital gain, where both are weighed relative to the original purchase price of stock (P_0).

Example 1.1.1 Gill purchases 50 shares of stock in a local firm at \$75.00 per share. He expects to get a dividend of \$4.00 per share at the end of the year. He also expects to sell his shares at \$80.00 each. What is his expected rate of return (r)?

$$r = \frac{D + P_1 - P_0}{P_0}$$

$$= \frac{\$4 + \$80 - \$75}{\$75}$$

$$= 12\%$$

We can also get this rate if we consider all shares.

$50 \times \$75 = \$3{,}750$	purchase price for investment
$50 \times \$80 = \$4{,}000$	sale price for investment
$50 \times \$4 = \$200$	total dividend expected

$$r = \frac{\$200 + \$4{,}000 - \$3{,}750}{\$3{,}750} = 12\%$$

Mathematically, we can obtain any variable in a formula in terms of the other variables available. Let's assume that we are given a forecast for next year's price of a stock (P_1), its dividend (D_1), and the market capitalization rate (r). Shouldn't that mean that we can calculate the current price of that stock (P_0)? Technically, yes. The current price (P_0) is

$$P_0 = \frac{D_1 + P_1}{1 + r}$$

Basically, this means that we discounted the future returns of the shares into their current value. We can also formulate the price next year (P_1) in terms of the expected dividend for the year after (D_2) and the price of that year (P_2):

$$P_1 = \frac{D_2 + P_2}{1 + r}$$

and if we plug this value of P_1 into the P_0 formula above, we get

$$P_0 = \frac{1}{1+r}(D_1 + P_1)$$

$$= \frac{1}{1+r}\left(D_1 + \frac{D2+P2}{1+r}\right)$$

$$= \frac{1}{1+r}\left[D_1 + \frac{1}{1+r}(D_2 + P_2)\right]$$

$$= \frac{D1}{1+r} + \frac{D2+P2}{(1+r)2}$$

and if we continue to get our estimation for the next year after (year 3), we get

$$P_0 = \frac{D_1}{1 + r} + \frac{D_2 + P_2}{(1 + r)^2} + \frac{D_3 + P_3}{(1 + r)^3}$$

and similarly for any number of years in the future, such as k, and if we substitute for P_2 in terms of P_3, and for P_3 in terms of P_4, and so on, all subsequent P 's would disappear and we end up with

$$P_0 = \frac{D_1}{1 + r} + \frac{D_2}{(1 + r)^2} + \cdot \cdot \cdot + \frac{D_k + P_k}{(1 + r)^k}$$

and the final summation would be

$$P_0 = \sum_{t=1}^{k} \frac{Dt}{(1+r)t} + \frac{Pk}{(1+r)k}$$

This means that the current price of a stock is actually equal to the sum of all discounted dividends for a number of future years, defined by the period k. It is noteworthy to say here that as k continues to increase and approaches infinity, the value of the last price would approach zero, and for this reason, the last price can be eliminated and the formula becomes

$$P_0 = \sum_{t=1}^{\infty} \frac{Dt}{(1+r)t}$$

This means that the present value of a common stock is just like the present value of any other asset. It is defined by the summation of the discounted stream of future dividends. However, this formula assumes a case of zero growth in dividends. In other words, the dividends stay constant throughout future years. It implies that all D's are the same, and therefore the formula can be written as

$$P_0 = \sum_{t=1}^{\infty} \frac{Dt}{(1+r)t}$$

and since $\sum_{t=1}^{\infty}[1/(1 + r)t]$ is just the table present value interest factor (PVIF) for any r and any t, we can rewrite the formula as

$$P_0 = D_1(\mathrm{PVIF}_{r,t})$$

$$\boxed{P_0 = \frac{D_1}{r}}$$

Example 1.1.2 If Bright Paint stock pays a \$14.95 dividend per share and it is expected to be constant indefinitely, what would the value of this stock be if the required return is $11\frac{1}{2}\%$?

$$P_0 = \frac{D_1}{r}$$

$$= \frac{\$14.95}{.115}$$

$$= \$130.00$$

But if the dividend grows in a constant rate such as g, the current value of the common stock would be adjusted by $D + Dg = D(1 + g)$:

$$P_0 = \frac{D_1(1+g)}{1+r} + \frac{D_2(1+g)^2}{(1+r)^2} + \cdots + \frac{D_k(1+g)^k}{(1+r)^k}$$

and we end up with

$$\boxed{P_0 = \frac{D_1}{r-g}} \qquad r > g$$

where r, the required return or the market capitalization rate, is assumed to be larger than g, the expected constant growth rate for dividends. This formula is called the **Gordon formula** after M. J. Gordon, who along with E. Shapiro, published an article entitled “Capital Equipment Analysis: The Required Rate of Profit,” which was published in *Management Science* in 1956. This formula was developed originally by J. B. Williams in his book *The Theory of Investment Value*, published by Harvard University Press in 1938, but stayed unpopular until Gordon and Shapiro rediscovered it 18 years later.

Since $D_1 = D_0 + D_0 g$, $D_1 = D_0(1 + g)$, we can write the foregoing formula as

$$\boxed{P_0 = \frac{D_0(1+g)}{r-g}}$$

Example 1.1.3 Suppose that an investor wishes to get a 12% yield from a stock whose estimated growth is 8%. Suppose also that the firm selling the stock has distributed 60% of its earnings as dividends, that its earnings per share is \$3.20, and that it sells the stock for \$40.00 per share. What would the value of the stock be, and what would be the expected return?

First, we get the dividend as 60% of the earnings per share.

$$D_0 = \$3.20(.60) = \$1.92$$

$$P_0 = \frac{D_0(1+g)}{r-g}$$

$$= \frac{\$1.92(1+.08)}{.12-.08}$$

$$= \$51.84 \qquad \text{what the current stock price should be}$$

$$r = \frac{D}{P_0} + g$$

$$= \frac{\$1.92}{\$40} + .08$$

$$= 12.8\%$$

Note that we used the actual price $40 for P_0, not the estimated current value ($51.84). Moreover, it is worthwhile to note that this r (the market capitalization rate that would often serve as the cost of common stock), would be obtained directly from the Gordon formula:

$$\boxed{r = \frac{D_1}{P_0} + g}$$

Example 1.1.4 The common stock for Big Book Company sells for $55 and its dividend in 2010 was $4.30, which has been growing through the last five years as shown in Table E1.2.4. What would the cost of this common stock be?

From the past history of dividends, we can calculate g, the rate of growth from 2005 to 2009:

$$g = \sqrt[n]{\frac{FV}{CV}} - 1$$

$$= \sqrt[4]{\frac{\$4.00}{\$2.95}} - 1$$

$$= .08$$

TABLE E1.2.4

Year	Dividend ($)
2009	4.00
2008	3.75
2007	3.33
2006	3.15
2005	2.95

$$r = \frac{D1}{P0} + g$$

$$= \frac{\$4.30}{\$55} + .08$$

$$= 15.8\ \%$$

There is another way to find g value. Given that D_1/P_0 is actually the dividend yield, g can be found by multiplying the return on equity (ROE) by the plowback ratio (Plow):

$$g = (\text{ROE})(\text{Plow})$$

where ROE is the ratio of the earnings per share (EPS) to the equity per share (Eq).

$$\text{ROE} = \frac{\text{EPS}}{\text{Eq}}$$

and Plow is the complementary amount of the payout ratio (Pout):

$$\text{Plow} = 1 - \text{Pout}$$

and the payout ratio (Pout) is the ratio of the expected dividend (D_1) to the earnings per share (EPS).

$$\text{Pout} = \frac{D_1}{\text{EPS}_1}$$

Example 1.1.5 A common stock is selling for $35.00 per share, and its expected dividend at the end of next year is $1.15 per share. The earning per share in this firm is $2.30, and the book equity per share is $14.50. What would the market capitalization rate or the cost of this stock be?

$$\text{ROE} = \frac{\text{EPS}}{\text{Eq}}$$

$$= \frac{\$2.30}{\$14.50} = 16\%$$

$$\text{Pout} = \frac{D_1}{\text{EPS}_1}$$

$$= \frac{\$1.15}{\$2.30} = .50$$

$$\text{Plow} = 1 - \text{Pout}$$

$$= 1 - .50 = .50$$

$$g = \text{ROE} \cdot \text{Plow}$$
$$= .16(.50) = .08$$
$$r = \frac{D1}{Po} + g$$
$$= \frac{\$1.15}{\$35} + .08$$
$$= 11.3\ \%$$

1.1.2 COST OF NEW ISSUES OF COMMON STOCK

The cost of a new issue of common stock is obtained by the same market capitalization rate formula but also by consideration of:

1. An underpriced stock to be sold at a price below its current market price (P_0). This may seem necessary to the firm in order to sell a new issue of the stock. The underpricing amount (Un) would be the difference between current price (P_0) and new price (P_n).

$$\text{Un} = P_0 - P_n$$

2. The flotation cost (FC), which is the cost of issuing and selling the new stock.

So if both items above are subtracted from the stock price, we would get what is called the **net proceed** (N_n), which would replace P_0 in the new formula of the cost of a new issue of stock (r_n).

$$\boxed{r_n = \frac{D}{N_n} + g}$$

Example 1.1.3 Suppose that there is a new issue of the stock in Example 1.2.5. The firm decides to underprice it by $1.55 per share and calculated the flotation cost at $.75 per share. Determine the cost of the new issue of this common stock.

$P_0 = 55$; Un = 1.55; Fc = \$.75; $D_1 = \$4.30$; $g = .08$.

$$N_n = P_0 - (\text{Un} + \text{Fc})$$
$$= 55 - (\$1.55 + \$.75)$$
$$= \$52.70$$
$$r_n = \frac{D}{N_n} + g$$
$$= \frac{\$4.30}{\$52.70} + .08$$
$$= 16.2\%$$

Note that the cost of the new issue is greater than the cost of the current stock because of dividing the dividend by the new net proceeds, which is less than the current price.

1.1.3 STOCK VALUE WITH TWO-STAGE DIVIDEND GROWTH

The Gordon formula assumes that the expected growth rate for dividend (g) is lower than the required return or market capitalization rate (r). But suppose that at some point, earnings and dividends go up to exhibit a very high growth, sufficient for g to be greater than r. In such a case, Gordon's formula would produce a negative value for the stock, and that is why we should estimate the value of stock using the two-stage growth formula, which addresses the growth in two stages and estimates the stock value by combining the present value of dividends from year 1 to n with the present value of dividends from $n + 1$ to infinity.

$$P_0 = D_0 . \frac{1 - [\,(1+g1)\,/\,(1+r)]n}{r - g1} \text{ (1+g1)+ Dn+1 } (\frac{1}{r-g2}) \,(\frac{1}{1+r})^n$$

Example 1.4.1 Find the current price per share of the common stock at SURE Corporation. Annual dividends are expected to grow by 25% for the next 5 years and slow back to 5% after that. Given that the dividend is $37.00 per share and the required rate of return is 15%.

$g_1 = .25$; $g_2 = .05$; $D_0 = \$37.00$; $r = .15$; $n = 5$.

$$P_0 = D_0 \,.\, \frac{1 - [(1+g1)\,/\,(1+r)]n}{r - g1} \,(\,1 + g1) + D_{n+1}\, (\frac{1}{r - g2}) \,\, (\frac{1}{1+r})^n$$

$$= 37\, [\frac{1 - [(\,1 + .25)\,/\,(1 + .15\,)]5}{.15 - .25}]\,(1 + .25)$$

$$= 37\,(1 + .25)5\,(1 + .05)\,\,(\frac{1}{.15 - .05})\,\,(\frac{1}{1 + .15})5$$

= \$239.23 + \$589.46 = \$828.69

1.1.4 COST OF STOCK THROUGH THE CAPM

The **capital asset pricing model** (CAPM) addresses the relationship between the required return (or in this context, the cost of common stock) and the nondiversifiable portion of the overall risk that a firm may face. The nondiversifiable risk is the external part of the asset's risk, which is attributable to out-of-firm circumstances and conditions and which affect all firms in the industry, therefore cannot be reduced or eliminated by the usual remedy of portfolio diversification. It is expressed in the model by beta (β) as an index to measure the variation of an asset's return in response to the changes in the market return. According to this model, the required return or cost of common stock (r_c) is obtained by

$$\boxed{r_c = R_f + \beta(R_m - R_f)}$$

The cost is actually a combination between the risk-free rate of return (R_f), which is measured by the return on a U.S. Treasury bill, and the risk premium $\beta(R_m - R_f)$ which is the difference between the market return (R_m) and the risk-free return (R_f) modified by beta (β).

Example 1.5.1 Use the CAPM to estimate the cost of Joyland's common stock if its beta equals 1.70 and the market return is 12%. Given the risk-free rate of 8%:

$$R_f = 8\%, R_m = 12\%, \text{ and } \beta = 1.70.$$

$$\begin{aligned} r_c &= R_f + \beta(R_m - R_f) \\ &= .08 + 1.70(.12 - .08) \\ &= 14.8\% \end{aligned}$$

1.1.5 OTHER METHODS FOR COMMON STOCK VALUATION

The P/E Multiples Method

In the **P/E multiples method**, the value of a common stock in a firm (V_c) would be measured according to use of the price/earning ratio (P/E_i) of the industry, which is often published by some standard financial reports. The value (V_c) would be obtained by multiplying the firm's expected earnings per share (EPS_e) by the industry's P/E multiples:

$$\boxed{V_c = \text{EPS}_e(P/E_i)}$$

This method is more suited for firms that are not publicly traded.

Example 1.6.1 A train service corporation expects its earnings per share to be \$3.10 at a time that the price/earnings ratio in the land transportation industry is rated at 9. What would be a good estimation of the value of a share in the train service corporation?

$$\begin{aligned} V_c &= \text{EPS}_e(P/E_i) \\ &= \$3.10(9) \\ &= \$27.90 \end{aligned}$$

The Book Value per Share Method

The **book value per share method**, is based on a hypothetical situation by which one would think that a good estimation of the value of a share in a firm would be the stockholder's per capita share of all the firm's net proceeds, if all assets are liquidated and all liabilities, including the preferred stocks, are paid. Assets and liabilities have their accounting value already recorded in the firm's books, which is why this method is called the **book value method**.

Example 1.6.2 The ABC Company's books show that the total assets are $2.7 million and total liabilities are $1.2 million. The company's preferred stock dividends would reach $650,000 and it has 85,000 common stock shares. What is the value of a share?

$$V_c = \frac{\$2{,}700{,}000 - (\$1{,}200{,}000 + \$650{,}000)}{85{,}000}$$

$$= \$10.00$$

The Liquidation Value per Share Method

The **liquidation value per share method** is exactly the same as the book value method except that this one is actual, not hypothetical. It is about the value of a share in a firm that is actually being liquidated.

Example 1.6.3 A firm's team responsible for handling the firm's assets liquidation reported that $2.3 million would remain after they pay all liabilities, including the preferred stock. The firm plans to distribute this net proceeds among the 110,000 shares of common stocks. The value of a share would, therefore, be

$$V_c = \frac{\$2{,}300{,}000}{110{,}000}$$

$$= \$20.91$$

1.1.6 VALUATION OF PREFERRED STOCK

Since preferred stock does not have any maturity date and it pays a fixed dividend for as long as the stock is outstanding, the payment of dividends can be considered a perpetuity, and for that we can write the formula of its value (P_p) as

$$P_p = D_p\left(\frac{1}{rp}\right)$$

$$\boxed{P_p = \frac{D_p}{r_p}}$$

Example 1.7.1 Wide Range Company pays a fixed annual dividend of $7.50 to their preferred stockholders. What is the value of this preferred stock if the required rate of return is estimated at 14.5%?

$$P_p = \frac{D_p}{r_p}$$
$$= \frac{\$7.50}{.145}$$
$$= \$51.72$$

1.1.7 COST OF PREFERRED STOCK

The **cost of preferred stock** is represented by the required rate of return (r_p), which can be obtained by rearranging the preceding formula:

$$r_p = \frac{D_p}{P_p}$$

But instead of using the value of the preferred stock (P_p), we have to use the net proceeds (N_p), which would be P_p after subtracting any flotation cost (FC).

$$N_p = P_p - \text{FC}$$

and the right formula becomes

$$\boxed{r_p = \frac{D_p}{N_p}}$$

Example 1.8.1 If XYZ Corporation pays out 9% of its stock par value of $96 as a dividend for preferred stock given that the cost to issue and sell the stock is $4.00, what is the cost of the stock?

$$D_p = .09(\$96) = \$8.64$$
$$N_p = \$96 - \$4 = \$92$$
$$r_p = \frac{D_p}{N_p}$$
$$= \frac{\$8.64}{\$92}$$
$$= 9.4\%$$

Bonds

Bonds are one of the major securities that are traded in the capital market. A **bond** is a long-term investment instrument that is usually issued and sold by big businesses and governments to a diverse group of lenders for the purpose of raising large amounts of capital. Bonds are written promises by the borrower (governments or corporations) to the lender (the investors, individuals or institutions) to pay:

1. The full amount borrowed on or before a certain date in the future called a **maturity date** or **redemption date**. This amount is also called the **redemption value**. It is often the amount that is printed on the face of the bond, called the **face value** or **par value** of the bond. It usually comes in a denomination of $1,000 or its multiples.
2. A fixed set of cash payoffs calculated as the semiannual interest on the face value, using a certain interest rate called the **bond rate**, **coupon rate**, or **contract rate**, and it is usually stated in annual form. These interest payments are paid throughout the maturity period and until the amount borrowed is redeemed.

The maturity period is often 10 years and more. It is usually divided into semiannual payments and the interest is paid on certain days called **payment days** or **coupon days**.

1.2 BOND VALUATION

Since the investor receives interest payments throughout the maturity period and receives the principal back on the redemption date, the value of the bond at the time of purchase is assessed as the present value of the future payments of interest plus the present value of the redemption amount. Suppose that a bond with a par value of $1,000.00 matures in 10 years, earning a 8% bond rate. The annual interest would be

$$\$1{,}000 \times .08 = \$80$$

But since the interest payments are made on semiannual terms, each payment would be $40 for the next 20 terms. The last term would be paying the 20th interest payment of $40 plus the original amount borrowed, $1,000: that is, $1,040.

Year 1		Year 2		−→	Year 9		Year 10		
term 1	term 2	term 3	term 4	−→	term 17	term 18	term 19	term 20	
40	40	40	40	−→	40	40	40	40	1,000

The value of this bond at the time of purchase *(B_0)* would be equal to the present value of all the future cash flows in the table above. So if we discount all interest payments and the redemption amount back to the current value, using a semiannual rate of .04 (.08/2) and a maturity of 20 *(*10 $\times$2*)*, we should get $1,000 as a current value:

$$B_0 = 40\left[\frac{1}{(1+.04)1}\right] + 40\left[\frac{1}{(1+.04)2}\right] + ...+ 40\left[\frac{1}{(1+.04)20}\right]$$

$$+1{,}000\left[\frac{1}{(1+.04)20}\right]$$

$$= 38.462 + 36.98 + ...+ 18.256 + 474.64$$

$$= \$1{,}000$$

Figure 2.1 shows how all cash flows of the bond, including the redemption value plus all the interest payments are brought back in time from the future to the present to form the discounted value in the current time. Suppose that the bond in the example is to be purchased in 2011 and to get redeemed in 2021. But if the investor wants to assess the value of this investment by comparing its return to returns of other alternatives, the comparison would require getting the present value of the same set of cash flows using an interest rate that differs from the bond rate stated. This interest rate is usually the prevailing rate of return on comparable investment. It is considered the desired or required rate of return (i). It is often called the **current rate** or the **yield to maturity** (YTM).

So let's assume that at the time of considering this bond as an investment purchase, the market rate of return on similar securities is 10%, which would be what the investor is giving up if buying the bond at 8%. In this case we can obtain the value of the bond of 8% coupon rate by discounting the previous cash flows at the yield rate of 10%, which gives us a semiannual rate of .05.

$$B0 = 40\left[\frac{1}{(1+.05)1}\right] + 40\left[\frac{1}{(1+.05)2}\right] + \ldots$$

$$+40\left[\frac{1}{(1+.05)20}\right] + 1{,}000\left[\frac{1}{(1+.05)20}\right]$$

$$= 38.09 + 36.28 + ... +15.07 + 376.90$$

$$= \$875.38$$

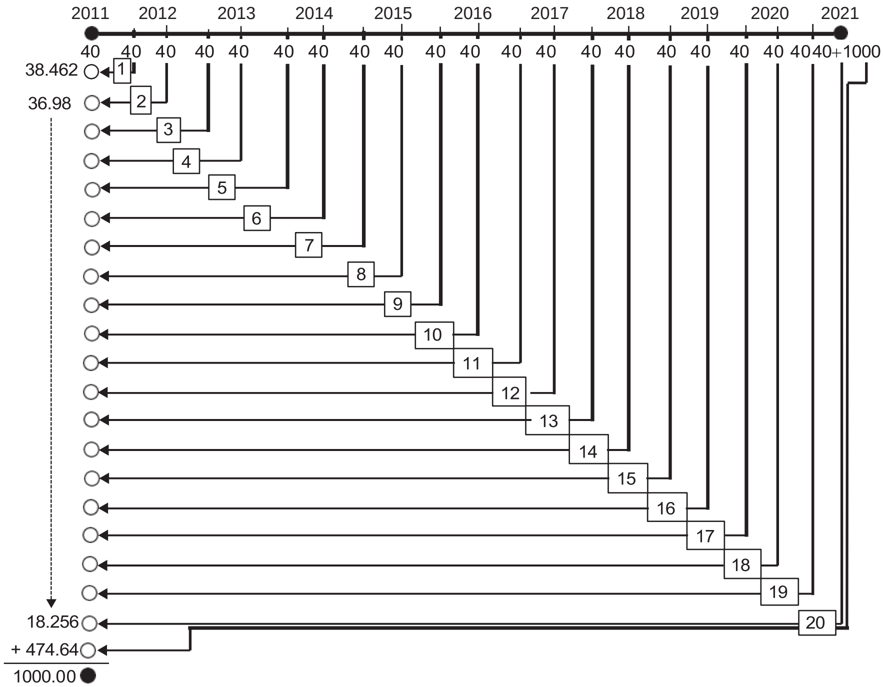

FIGURE 2.1

Notice that the value of the bond turned out to be less than before when using a higher rate of return. The reason is that we are reversing the direction from the future to the present when we discount. The more money grows forward, the more it shrinks backward.

We can use the following general formula to obtain the value of a bond with a required yield rate (i):

$$B_0 = I \left[\sum_{t=1}^{n} \frac{1}{(1+i)t} \right] + M \frac{1}{(1+i)n}$$

where B_0 is the value of the bond or the purchase price at the required or desired yield rate, I is the interest payment per semiannual term, i is the required or desired interest yield rate, t is any semiannual term, n is the number of semiannual terms in the maturity period, and M is the redemption value of the bond.

Knowing that the term $\sum_{t=1}^{n} 1/(1+i)^t$ is the accumulated present value interest factor (PV/FA), that the term $1/(1+i)^n$ is the present value interest factor (PVIF), and that both of them are the same table values that we dealt with before, we can rewrite the formula above in the following way, which would allow us to use the table values:

$$\boxed{B_0 = I(\text{PV/FA}_{i,n}) + M(\text{PVIF}_{i,n})}$$

and it can also be written, using $a_{\overline{n}|i}$ and v^n, as we have shown earlier. The formula can then also be written this way:

$$\boxed{B_0 = I(a_{\overline{n}|i}) + M(v^n)}$$

The more the semiannual terms, the more tedious the manual calculations become and therefore utilizing the table values comes in handy and makes it much easier.

Example 2.1.1 A $1,000 bond is maturing in 15 years. Its semiannual coupon is at $9\frac{1}{2}$%. Find the purchase price that would yield 11%.

The annual coupon rate is $9\frac{1}{2}$%, so the semiannual rate is $4\frac{3}{3}$% or .0475. $I = \$1{,}000(.0475) = \$47.50$. The redemption value ($M$) is $1,000, the annual yield is 11%, and the semiannual yield is $5\frac{1}{2}$% or .055.

$$\begin{aligned} B_0 &= I(a_{\overline{n}|i}) + M(v^n) \\ &= \$47.50(a_{\overline{30}|.055}) + \$1{,}000(v^n) \\ &= \$47.50(14.534) + \$1{,}000(.2006) \\ &= \$890.96 \end{aligned}$$

Example 2.1.2 If the face value of a bond is $2,000, its coupon rate is $12\frac{1}{4}$%, and if it is redeemable at par at the end of 10 years, what would its current value be if the investor wants the yield to be 10%?

The bond rate is $12\frac{1}{4}$% annually, so the semiannual rate is .06125. The interest payment would be

$$\$2{,}000 \times .06125 = \$122.50$$

The redemption value (M) is $2,000, the annual required rate of return is 10% and the semiannual rate is 5%.

$$\begin{aligned} B_0 &= I(a_{\overline{n}|i}) + M(v^n) \\ &= \$122.50(a_{\overline{20}|.05}) + \$2{,}000(v^n) \\ &= \$122.50(12.462) + \$2{,}000(.3769) \\ &= \$2{,}280.40 \end{aligned}$$

1.2.1 PREMIUM AND DISCOUNT PRICES

In Examples 2.1.1 and 2.1.2 we saw that when the required yield was larger than the bond rate, the value or purchase price was lower than the face value of the bond, and when the yield rate was lower than the bond rate, the value or purchase price was higher than the face value.

$$\text{if} \quad i > r: \ B_0 \downarrow$$

$$\text{if} \quad i < r: B_0 \uparrow$$

If the \$1,000 bond at $9\frac{1}{2}$% bond rate is sold for \$890.96 to yield 11% return, it is said to be sold at discount, and the discount amount is the difference between the face value and the purchase price:

$$\$1{,}000 - \$890.96 = \$109.04 \qquad \text{discount}$$

Similarly, if the \$2,000 bond at a $12\frac{1}{4}$% bond rate is sold for \$2,280,40 to give a 10% yield, the bond is said to be sold at premium, and the premium amount would be the difference between the purchase price and the face value:

$$\$2{,}280.40 - \$2{,}000 = \$280.40 \qquad \text{premium}$$

So when the purchase price is different from the face value of the bond, the difference, discount or premium, is carried by the buyer (investor). We can look at the discount of \$109.04 this way:

$$\frac{\$1{,}000 \times .095}{2} = \$47.50 \qquad \text{semiannual interest based on } 9\tfrac{1}{2}\%$$

and

$$\frac{\$1{,}000 \times .11}{2} = \$55.00 \qquad \text{semiannual interest based on 11\%}$$

$$\$55.00 - \$47.50 = \$7.50 \qquad \text{difference in interest per term}$$

If we discount this difference for the entire maturity (30 terms) at a yield rate of 11% (semiannual 5.5%), we get

$$\$7.50(a_{\overline{30|}.055})$$

$$47.50(14.534) = \$109.04 \qquad \text{discount}$$

which says that the discount is the present value of the differences in interest throughout the entire maturity period and that it must be subtracted from the par value (redemption amount) to form the current value or purchase price of the bond:

$$M - D_s = B_0$$

$$\$1{,}000 - \$109.04 = \$890.96$$

In the same manner, we can look at the premium of \$280.40 in the following way:

$$\frac{\$2{,}000 \times .1225}{2} = \$122.50 \qquad \text{semiannual interest based on } 12\tfrac{1}{4}\%$$

$$\frac{\$2{,}000 \times .10}{2} = \$100.00 \qquad \text{semiannual interest based on } 10\%$$

$$\$122.50 - \$100 = \$22.50 \qquad \text{difference in interest per term}$$

Discounting this difference at a yield rate of 10% or 5% semiannually for 20 terms gives us

$$\$22.50(a_{\overline{20}|.05})$$

$$\$22.50(12.462) = \$280.40 \qquad \text{premium}$$

This also says that the premium is, in fact, the present value of the difference in interest throughout the maturity period of 20 terms and therefore must be added to the full (redemption) value to form the purchase price of the bond:

$$M + P_m = B_p$$

$$\$2{,}000 + \$280.40 = \$2{,}280.40$$

Generally, we can write the formula of discount (D_s) and premium (P_m) as the present value of the differences between face value and the purchase price of a bond:

$$D_s = (M_i - M_r)a_{\overline{n}|i}$$

$$\boxed{D_s = M(i - r)a_{\overline{n}|i}} \qquad \text{when} \quad i > r$$

$$P_m = (M_r - M_i)a_{\overline{n}|i}$$

$$\boxed{P_m = M(r - i)a_{\overline{n}|i}} \qquad \text{when} \quad i < r$$

In conclusion, a bond may sell at a value that is less than its par value when the required rate of return is greater than the rate stated on the bond. It would be a case of selling at a discount. It may also sell at a value higher than its face value when the required rate of return is lower than the bond rate. That would be selling at a premium.

Example 2.2.1 A bond with a face value of \$5,000 is redeemable in 8 years at a bond rate of 7.5%. If it is to be purchased to yield 6%, would it be purchased at a premium or discount, and what would the purchase price be?

First let's get the semiannual rates:

$$\text{semiannual bond rate} = .075/2 = .0375$$

$$\text{semiannual yield} = .06/2 = .03$$

Since the required yield is less than the bond rate, the bond would sell at a premium.

$$\begin{aligned} P_m &= M(r - i)a_{\overline{n}|i} \\ &= \$5{,}000(.0375 - .03)a_{\overline{8}|.03} \\ &= \$5{,}000(.0075)(7.019) \\ &= \$263.24 \qquad \text{premium} \\ B_p &= M + P_m \\ &= \$5{,}000 + \$263.24 = \$5{,}263.24 \end{aligned}$$

Example 2.2.2 A \$2,000 bond pays $10\frac{1}{2}$% bond interest twice a year for 20 years, at the end of which it will be redeemable at par value. If the investor wants to get a 12% yield, would the bond sell at a premium or a discount, and for how much?

$$\text{semiannual bond rate} = .105/2 = .0525$$

$$\text{semiannual yield} = .12/2 = .06$$

Since the yield is higher than the coupon rate, the bond would sell at a discount.

$$\begin{aligned} D_s &= M(i - r)a_{\overline{22}|.06} \\ &= \$2{,}000(.06 - .0525)(11.4699) \\ &= \$172.05 \qquad \text{discount} \\ B_d &= M - D_s \\ &= \$2{,}000 - \$172.05 = \$1{,}827.95 \end{aligned}$$

1.2.2 PREMIUM AMORTIZATION

As mentioned earlier, a premium is recovered gradually throughout the interest payments, and therefore the investor would not receive the premium price at redemption. Only the original redemption value would be received in the final term. For that reason the recorded premium price on the investor's book should be reduced as the fraction of premium is received semiannually. In other words,

TABLE 2.1 Premium Amortization Schedule

(1) Semiannual Term	(2) Interest Payment [(par) (r)]	(3) Yield .0225 [.0225(5)]	(4) Amortization of Premiums [(2) − (3)]	(5) Book Value of the Bond [(5) − (4)]
				3,163.06
1	90	71.17	18.83	3,144.23
2	90	70.74	19.25	3,124.97
3	90	70.31	19.69	3,105.28
4	90	69.87	20.13	3,085.15
5	90	69.42	20.58	3,064.57
6	90	68.95	21.05	3,043.52
7	90	68.48	21.52	3,022.00
8	90	67.99	22.00	3,000.00
	720	556.94	163.06	

the book value of the premium price has to be amortized and reduced to be equal to the original redemption value when the redemption date comes.

Let's consider a bond of \$3,000 at 6% redeemable in 4 years and purchased to provide a $4\frac{1}{2}$% yield. The semiannual bond rate is .03 and the semiannual yield is .0225.

$$\begin{aligned}
P_m &= M(r - i)a_{\overline{n}|i} \\
&= \$3{,}000(.03 - .0225)a_{\overline{8}|.0225} \\
&= \$3{,}000(.0075)(7.2472) \\
&= \$163.06 \\
B_p &= \$3{,}000 + \$163.06 = \$3{,}163.06
\end{aligned}$$

The amortization schedule in Table 2.1 shows how the premium price of \$3,085.16 is gradually reduced to the redemption value of \$3,000 by the end of the 8th semiannual term of the maturity time of 4 years.

$$\mathrm{I} = \$3{,}000(.03) = \$90$$

To construct the amortization table we carried out the following steps:

1. The first entry was the premium price \$3,163.06 as the first book value.
2. We multiplied the book value by the yield rate to get the first entry in column (3):

$$\$3{,}163.08 \times .0025 = \$71.17$$

3. We subtracted \$71.17 from the interest of the first term (90) to get the first entry on column (4):

$$\$90 - \$71.17 = \$18.83$$

4. We got the second entry of the book value by subtracting the amortization amount of \$18.83 from the previous book value:

$$\$3{,}163.06 - \$18.83 = \$3{,}144.23$$

5. We multiplied this new book value of \$3,144.23 by the yield rate as in step 2 and repeat the sequence until you get the following results:
   (a) The last book value must equal the face value of the bond: in this case, \$3,000.
   (b) The total of column (4) must equal the premium: in this case, \$163.06.
   (c) The accumulated yield [total of column (3)] must equal the difference between the total interest and the premium: in this case:

$$\$720 - \$163.06 = \$556.94.$$

1.2.3 DISCOUNT ACCUMULATION

Unlike the case of the premium price, the discount price is less than the face value, but the investor would still receive the face value at redemption. Therefore, the total deficit of the discount would be recovered gradually and little by little through the interest payments. This would require the book value to increase gradually from the discount price until it becomes equal to the face value of the bond on the redemption day, and that is what the discount accumulation is all about.

Let's consider a bond whose face value is \$8,000 redeemable in 3 years at 5%, but it is purchased to yield 8%.

$$\text{semiannual bond rate} = .05/2 = .025$$

$$\text{semiannual yield} = .08/2 = .04$$

Since the yield is greater than the bond rate, the bond is purchased at discount.

$$\begin{aligned} D_s &= M(I - r)a_{\overline{n}|i} \\ &= \$8{,}000(.04 - .025)a_{\overline{6}|.04} \\ &= \$8{,}000(.015)(5.2421) \\ D_s &= \$629.05 \qquad \text{discount} \\ B_d &= \$8{,}000 - \$629.05 = \$7{,}370.95 \qquad \text{purchase price} \end{aligned}$$

TABLE 2.2 Discount Accumulation Schedule

(1) Semiannual Term	(2) Interest Payment [(par) (r)]	(3) Yield .04 [.04(5)]	(4) Accumulation of Discount [(3) − (2)]	(5) Book Value of the Bond [(4) + (5)]
				7,370.95
1	200	294.84	94.84	7,465.79
2	200	298.63	98.63	7,564.42
3	200	302.58	102.58	7,666.99
4	200	306.68	106.68	7,773.67
5	290	310.95	110.95	7,884.62
6	200	315.38	115.38	8,000.00
	1,200	1,829.05	629.05	

$$
\begin{aligned}
I &= \$8{,}000(.025) \\
&= \$200 \qquad \text{semiannual interest payment}
\end{aligned}
$$

To construct Table 2.2 we carried out the following steps:

6. The first entry was the discount price $7,370.95 as the first recording of the book value.
7. We multiplied the book value by the yield rate to get the first entry in the column (3).

$$\$7{,}730.95 \times .04 = \$294.84$$

8. We subtracted the interest of the first term ($200) from $294.84 to get the first entry in column (4) ($94.84).

$$\$294.84 - \$200 = \$94.84$$

9. We added this $94.84 to the first book value to get the second book value.

$$\$7{,}370.95 + \$94.84 = \$7{,}465.79$$

10. We multiplied this new book value by the yield rate as in step 2 and repeated steps 2 to 5 until all the deficits were accumulated to complete the book value in the face value of $3,000.00 as the last recorded book value.
11. We checked that:
    (a) The total of column (4) is equal to the amount of the discount ($629.05).
    (b) The total of column (3) is the sum of the discount and the total interest.

$$\$1{,}829.05 = \$629.05 + \$1{,}200$$

1.2.4 BOND PURCHASE PRICE BETWEEN INTEREST DAYS

The bond price (B_0) that we established earlier assumed that an investor would buy on one of the interest payment dates of the maturity time. But, in reality, that may or may not happen. In case the purchase occurs on any day in between these established coupon days, we need to know two more bond price terms: purchase price between dates (B_{bd}), sometimes called the **flat price**, and the **quoted price** (B_q). We also need to distinguish between the interest at the bond rate and the interest at the yield rate, both of which are involved in the calculation.

$$\boxed{B_{bd} = B_0 + Y_{bp}}$$

So the bond purchase price between dates (B_{bd}) is a sum of the purchase price (B_0) calculated on the day of payment that immediately precedes the day of purchase, and the portion of interest (Y_{bp}) on the purchase price (B_0) and for the time between the purchase date and the day of payment before it (bp). This interest is calculated based on the yield rate (i):

$$Y_{bp} = B_0(i)\mathrm{b_p}$$

The quoted price (B_q), sometimes called the **net price**, is

$$\boxed{B_q = B_{bd} - I_{ac}}$$

where I_{ac} is the interest on the face value (M) and for the time between the purchase date and the payment date before it (bp), but calculated based on the bond or coupon rate (r). It is called the **accrued interest** and represents the interest portion that belongs to the seller.

$$I_{ac} = M(r)\mathrm{b_p}$$

Example 2.5.1 A bond of \$1,000 at 8% maturing in 10 years on February 2020 (see Figure E2.5.1). Its interest has been paid every August 15 and February 15 starting August 15, 2010. Find the purchase price and the quoted price if the bond is purchased on December 15 to make an 11% yield.

The semiannual bond rate = .08/2 = .04; the yield rate = .11/2 = .055; $n = 20$; $M = \$1{,}000$; $I = \$1{,}000(.04) = 40$.

$$\begin{aligned} B_0 &= I(a_{\overline{n}|i}) + M(v^n) \\ &= 40(a_{\overline{n}|i}) + \$1{,}000(v^n) \\ &= 40(11.9504) + \$1{,}000(.3427) \\ &= \$820.72 \qquad \text{purchase price on the interest date} \end{aligned}$$

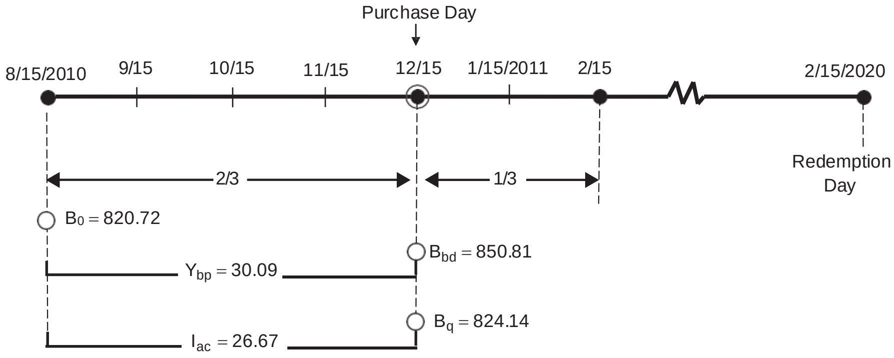

FIGURE E2.5.1

$$
\begin{aligned}
Y_{\mathrm{bp}} &= B_0(i)\mathrm{b_p} \\
&= \$820.72(.055)(\tfrac{1}{3}) \\
&= \$30.09 \\
B_{\mathrm{bd}} &= \$820.72 + \$30.09 \\
&= \$850.81 \\
I_{\mathrm{ac}} &= M(r)\mathrm{b_p} \\
&= \$1{,}000(.04)(\tfrac{1}{3}) \\
&= \$26.67 \qquad \text{accrued interest} \\
B_q &= B_{\mathrm{bp}} - I_{\mathrm{ac}} \\
&= \$850.81 - \$26.67 \\
&= \$824.14
\end{aligned}
$$

Example 2.5.2 A bond of \$5,000 at 5% matures in 6 years and 4 months (see Figure E2.5.2). What would the flat price be to yield 7%? Also find the seller's share of the accrued interest and the bond net price.

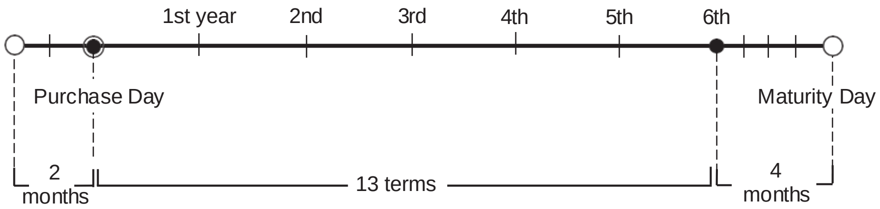

FIGURE E2.5.2

The flat price is the purchase price between dates (B_{bd}), the seller's share is the accrued interest (I_{ac}), and the net price is the price quoted (B_q). Since the redemption date is 4 months after the sixth year, we can conclude that the interest date immediately preceding the purchase date would have been 2 months before the purchase date. We can conclude that the maturity term is 13, which is 6 years × 2 terms, and the 4 months are written as the 13th term.

M = \$5,000; r = 5%; semiannual rate = .05/2 = .025; i = .07; semiannual rate = .07/2 = .035; n = 13 terms; bp = 2months = $\frac{1}{3}$ of a term.

$$
\begin{aligned}
I &= M(r) \\
&= \$5{,}000(.025) = \$125 \\
B_0 &= I\,(a_{\overline{13}|.035}) + M(v^n) \\
&= \$125(10.3027) + \$5{,}000(.6394) \\
&= \$4{,}484.84 \qquad \text{purchase price on the payment date} \\
Y_{bp} &= B_0(i)\,\mathrm{b_p} \\
&= \$4{,}484.84(.035)\left(\tfrac{1}{3}\right) \\
&= \$52.32 \\
B_{bd} &= B_0 + Y_{bp} \\
&= \$4{,}484.84 + \$52.32 \\
&= \$4{,}537.16 \qquad \text{purchase price between dates(flat price)} \\
I_{ac} &= M(r)\,\mathrm{b_p} \\
&= \$5{,}000(.025)\left(\tfrac{1}{3}\right) \\
&= \$41.67 \qquad \text{accrued interest or seller's share} \\
B_q &= B_{bd} - I_{ac} \\
&= \$4{,}537.15 - \$41.67 \\
&= \$4{,}495.49 \qquad \text{net or quoted price}
\end{aligned}
$$

Note that the price between dates (B_{bd}) can also be obtained by what is called the **practical method**, which is based on a simple interest formula:

$$\boxed{B_{bd} = B_0[1 + i(\mathrm{bp})]}$$

where bp is the portion of time before purchase, as we have seen. In our example, this should give us the same result:

$$
\begin{aligned}
B_{bd} &= \$4{,}484.84\left[1 + (.035)\left(\tfrac{1}{3}\right)\right] \\
&= \$4{,}537.16
\end{aligned}
$$

1.2.5 ESTIMATING THE YIELD RATE

Bonds for sale in the investment market are usually listed at their quoted or net price. As we have seen before, investors can calculate the value of the prospective bond at the yield rate desired, mainly to compare to the price quoted. If the value they calculate is greater or at least equal to the quoted price, they would know that their purchase would bring a return equal to or greater than the desired yield rate. If the yield is not known, it can be approximated for the purpose of that comparison. Three methods can be used to approximate the yield rate.

The Average Method

The **average method**, also called the **bond salesman's method**, uses the average investment as the base of comparison for the annual interest income. So the yield rate (YR) is obtained by dividing the annual interest income (AII) by the average annual investment (AAI).

$$\boxed{\text{YR} = \frac{\text{AII}}{\text{AAI}}}$$

Annual interest income is obtained by applying the annual bond rate (r) on the face value (M) and adjusting it for the premium or discount in their average sense. The average premium (P_m/n) would be subtracted and the average discount (D_s/n) would be added.

$$\text{AII} = M(r) - \frac{P_m}{n}$$

or

$$\text{AII} = M(r) + \frac{D_s}{n}$$

The **average annual investment** (AAI) is the average between the face value (M) and the price quoted, B_q:

$$\text{AAI} = \frac{M + B_q}{2}$$

Example 2.6.1 If a $4,000 at 5.5% coupon rate is to be purchased at a quoted price of $4,350 to be redeemed in 14 years, what is the yield rate?

$$\begin{aligned} P_m &= B_q - M \\ &= \$4{,}350 - \$4{,}000 = \$350 \end{aligned}$$

$$\begin{aligned}
\text{AII} &= M(r) - \frac{P_m}{n} \\
&= \$4{,}000(.055) - \frac{\$350}{14} \\
&= \$195 \\
\text{AAI} &= \frac{M + B_q}{2} \\
&= \frac{\$4{,}000 + \$4{,}350}{2} \\
&= \$4{,}175 \\
\text{yield rate}(\text{YR}) &= \frac{\text{AII}}{\text{AAI}} \\
&= \frac{\$195}{\$4{,}175} \\
&= .0467 = 4.67\%
\end{aligned}$$

It is noteworthy to mention that the average annual investment can be obtained by getting the actual amounts of investments which are connected to either the premium price or the discount price in an arithmetic progression whose common difference is the average premium (P_m/n) or the average discount (D_s/n). So, to illustrate, we can get the investment amounts in Example 2.6.1 starting with the premium price of \$4,350 and reducing it by the average premium $\$350/14 = 25$, down to the face value of \$4,000. Therefore, the arithmetic progression of investment would be \$4,350, \$4,325, \$4,300, \$4,275, \$4,250, \$4,225, \$4,200, \$4,175, \$4,150, \$4,125, \$4,100, \$4,075, \$4,050, \$4,025, \$4,000. The actual average of these amounts is their total divided by their number:

$$\frac{\$62{,}625}{15} = \$4{,}175$$

The Interpolation Method

According to the mathematical **interpolation method**, the yield rate can be approximated by association and comparison with related variables. We start with two close yield rates, use them to calculate their associated purchase price, and then set up an interpolation table and solve for the unknown yield rate.

Example 2.6.2 Let’s try to use Example 2.6.1 to approximate the yield rate by interpolation. We have a quoted price of \$4,350 that is a premium price. It indicates immediately that the yield rate would be lower than the bond rate of 5.5% or 2.75% semiannually. If we look at the PVIFA table, we would realize that the closest values of rates on the table are 2.5% and 2.25%. Let's use those rates to get their associated purchase premium prices:

$$P_m = M(r - i)a_{\overline{n}|i}$$

$$B_p = M + P_m$$

For 2.5% with $n = 28$ and $M = \$4000$:

$$\begin{aligned}
P_m &= \$4{,}000(.0275 - .0225)a_{\overline{28}|.0225} \\
&= \$4{,}000(.005)(20.6078) \\
&= \$412.16 \\
B_p &= \$4{,}000 + \$412.16 \\
&= \$4{,}412.16
\end{aligned}$$

For 2.25% with $n = 28$ and $M = \$4000$:

$$\begin{aligned}
P_m &= \$4{,}000(.0275 - .025)a_{\overline{28}|.025} \\
&= \$4{,}000(.0025)(19.9648) \\
&= \$199.65 \\
B_p &= \$4{,}000 + \$199.65 \\
&= \$4{,}199.65
\end{aligned}$$

Now we have three prices (\$4,412.16, \$4,350.00, \$4,199.65) and two rates (2.25% and 2.5%). We should be able to solve for the missing rate by setting the interpolation table:

		Yield Rate (%)	PurchasePrice (\$)		
2.25 – 2.25	2.25 – x	2.25	4412.16	4412.16 – \$4350.00	4412.16 – \$4199.65
		x	4350.00		
		2.5	4199.65		

$$\frac{2.25 - x}{2.25 - 2.5} = \frac{\$4{,}412.16 - \$4{,}350.06}{\$4{,}412.16 - \$4{,}199.65}$$

$$\frac{2.25 - x}{-.25} = \frac{62.16}{212.51}$$

$$\$478.15 - \$212.51x = -\$15.54$$
$$493.69 = 212.51x$$
$$\frac{493.69}{212.51} = x$$
$$\boxed{2.32 = x}$$

So the approximate yield is 2.32%. In fact, if we use it to get the purchase price, and if we use a table value that is in between the two readings above, we get a price very close to $4,350. With a sophisticated computer program, the exact yield can be found by the touch of a button.

The Current Yield Method

The **current yield method** approximates the yield rate based on a calculated rate of interest (Cr) which is obtained by dividing the interest (I) by the quoted price (B_q), then using Cr to get the approximate yield rate (YR):

$$\boxed{\text{YR} = \text{Cr}(2 + \text{Cr})}$$

where Cr is

$$\text{Cr} = \frac{I}{B_q}$$

Note that in case of the premium price, the average periodic premium P_m/n has to be subtracted from the interest.

$$\text{Cr} = \frac{I - P_m/n}{B_q}$$

Example 2.6.3 If we run this method over Example 2.6.2, where the premium price was $4,350, the face value was $4,000, and the bond rate was 5.5% for 14 years, we should obtain the average premium first:

$$P_m = \$350$$
$$\frac{P_m}{n} = \frac{\$350}{28} = \$12.50$$

which has to be taken off the interest I:

$$I = \$4{,}000(.055) = \$220$$
$$I - \frac{P_m}{n} = \$220 - \$12.50 = \$207.50$$

$$\text{Cr} = \frac{I}{B_q}$$

$$= \frac{\$207.50}{\$4,350} = .048$$

$$\text{YR} = \text{Cr}(2 + \text{Cr})$$

$$= .048(2 + .048)$$

$$= .098 \qquad \text{annual rate}$$

and the semiannual rate is .049.

1.2.6 DURATION

Duration is the average maturity of cash flows on an investment weighted by the present value of the cash flows. It is a measure of the reinvestment rate risk, specifically gauging how sensitive bond prices are to changes in interest rates. It is expressed by

$$D = \frac{I\ \sum_{t=1}^{n}[\frac{t}{(1+i)t}] + \frac{nM}{(1+i)n}}{I\ \sum_{t=1}^{n}[\frac{1}{(1+i)t}] + \frac{M}{(1+i)n}}$$

where I is the annual interest on a bond, t is the payment term, n is the number of years to redemption, i is the yield, and M is the par value of the bond.

Example 2.7.1 Consider a bond with face value \$1,000 at a bond rate of 9% paying annually and redeemable in 5 years. Given that the market yield is 12%, calculate the duration of this bond and explain its effect.

$t = 1, 2, 3, 4, 5;\ n = 5;\ i = .12;\ M = \$1,000.$

$$I = M(r)$$

$$= \$1,000(.09) = \$90$$

$$D = \frac{I\ \sum_{t=1}^{n}[\frac{t}{(1+i)t}] + \frac{nM}{(1+i)n}}{I\ \sum_{t=1}^{n}[\frac{1}{(1+i)t}] + \frac{M}{(1+i)n}}$$

$$= \frac{\$90\ (10) + (5 \times 1,000)\ /\ (1+.12)5}{\$90\ (3.61\ ) + (1,000)\ /\ (1+.12)5}$$

TABLE E2.7.1

Terms of Payments t	$\frac{t}{(1+i)^t}$	$\frac{1}{(1+i)^t}$
1	.893	.893
2	1.595	.797
3	2.135	.712
4	2.542	.636
5	2.837	.567
	10.00	3.61

$$D = \frac{\$3{,}737.13}{\$892.33}$$

$$= \$4.19$$

Duration is an element in gauging the sensitivity of bond price to any change in the yield rate.

$$\boxed{\%.\Delta B = \frac{\Delta i \cdot D}{1+i}}$$

Let's suppose that the yield has increased from 12% to 12.1%. The percentage change in bond price $(\%.6.B)$ would be

$$\%\ \Delta B = \frac{.1\ (\$4.19)}{1+.12}$$

$$= \$.37$$

which means that as the yield changes by .1, the bond price changes by \$.37.

The term $D/(1+i)$ called the **volatility** (VL), which leads us to consider the last formula and rewrite it as

$$\boxed{\%\Delta B = \Delta i \cdot VL}$$

where volatility is defined as a measure of how briskly the present value of cash flows respond to a change in the market interest rate.

Mutual Funds

Mutual funds are not specific securities such as stocks or bonds. They are actually a financial intermediary that pools funds of investors and makes them available for various investment opportunities by businesses and governments. Most common is the purchase of large blocks of stocks and bonds and other investment instruments. The most striking feature of mutual funds is the creation of a diversified portfolio of investment securities that is professionally managed and monitored. Each fund has its own specific investment goals and risk tolerance to reflect the individual objectives and character of the investors. Mutual fund companies claim a unique role in providing high-quality services to their investors, characterized by:

- Minimizing the levels of unsystematic risk by instituting the formal disciplined diversification of investment instruments in the face of transaction costs.
- Providing a high-standard around-the-clock level of professional management that has been able to deliver high rates of return and successful predictions of future trends and price charges.

Mutual funds are tailored to fit the need, nature, and specific objectives of various investors. They come into four major categories:

1. *Funds for income*. Their major objective is to provide a stable level of income. They are focused primarily on corporate and government bonds.
2. *Funds for growth*. The major objective of these funds is capital appreciation. They focus on the common stock of publicly held corporations. They vary in their aggressiveness and risk levels.
3. *Balanced funds*. These funds provide a balance of income and appreciation. They provide a fixed income as well as seeking capital growth opportunities. They invest in both stocks and bonds and are very popular with the large sector of investors who have a moderate level of risk tolerance.
4. *Global funds*. These are basically balanced funds, but they focus on investment opportunities abroad.

1.3 FUND EVALUATION

Unlike stocks, which are traded throughout the entire business day, a mutual fund's value is determined only at the end of each trading day. They are priced based on what is called **net asset value** (NAV), which would settle after the market has been closed. The net asset value is the mutual fund's equivalent of share price. It usually stands for the ability of the fund's management to deliver continuous and consistent profits. It is also a direct indicator of the market value. Net asset value consists of the total value of the holdings of the fund (market value and cash) after subtracting any liabilities or obligations (O). Dividing by the number of shares outstanding, we obtain the net asset value per share:

$$\boxed{\mathrm{NAV} = \frac{1}{S}[(\mathrm{MV} + C) - O]}$$

where NAV is the net asset value per share, MV is the market value of assets invested, C is the cash on hand, S is the number of shares outstanding, and O is the obligations or liabilities.

Example 3.1.1 The Bright Future Mutual Funds Company owns the following shares in four major securities:

Security 1: 333,200 shares

Security 2: 298,513 shares

Security 3: 197,814 shares

Security 4: 88,500 shares

The company also holds \$244,000 in cash, is responsible for \$153,000 in liabilities, and has a total of 395,667 shares outstanding. Calculate the fund's net asset value on a day where the share price for each security is \$12, \$16, \$20, and \$22, respectively.

First we calculate the market value for four securities at four prices:

$$\begin{aligned} \mathrm{MV} &= (333{,}210 \times 12) + (298{,}513 \times 16) + (197{,}814 \times 20) \\ &\quad + (88{,}500 \times 22) \\ &= 14{,}678{,}008 \end{aligned}$$

$$\begin{aligned} \mathrm{NAV} &= \frac{1}{S}[(\mathrm{MV} + C) - L] \\ &= \frac{1}{395{,}667}[(14{,}678{,}008 + 244{,}000) - 153{,}000] \\ &= \$37.33 \end{aligned}$$

1.3.1 LOADS

Loads are the commission or transaction fees imposed on the purchase and sale of mutual funds. Not all mutual funds carry these loads. In fact, they are named based on their inclusion as fees. A **load fund** is a fund that requires investors, buyers, and sellers of funds to pay these charges either per transaction, or as a percentage of return, or both. A **front-end load** is a charge imposed on the purchase transaction and a **back-end load** is a charge imposed on the sale transaction. It is also called a **deferred sales charge** or **contingent commission**. Loads range in value between 1% and 10% (at most) on the investment amount. **No-load funds** do not require paying these transaction charges but may include other types of charges. Factoring those charges in, we obtain the purchase price (PP) and the selling price (SP), which comprise the net asset value adjusted for the front-end load (L_F) and the back-end load (L_B).

$$PP = NAV\left(\frac{1}{1 - LF}\right)$$

$$SP = NAV\,(1 - L_B)$$

Example 3.2.1 If the front - end load is 6½ % and the back - end load is 5¼ %, what is the previous net asset value as to the offering and selling prices?

$$PP = NAV\left(\frac{1}{1 - LF}\right)$$

$$= \$37.33\ \left(\frac{1}{1 - .065}\right)$$

$$= \$39.93$$

$$SP = NAV(1 - L_B)$$

$$= \$37.33(1 - .0525)$$

$$= \$35.37$$

1.3.2 PERFORMANCE MEASURES

The Expense Ratio (ER)

The **expense ratio** is a performance measure especially useful for comparing the cost of investing between two or more funds. It reflects a fund's operating expenses as they are relative to the average asset throughout the year. It is therefore obtained by dividing the total expenses charged by the average of net assets of the fund.

$$\text{ER} = \text{Exp}\left[\frac{1}{(A1 + A2)/2}\right]$$

Example 3.3.1 Suppose that a mutual fund company shows its total assets at the start of the year as \$3,950,000, and at the end of the year as \$3,873,150. Suppose that the fund's total operating expenses have reached \$35,000. What would the expense ratio be?

$$\text{ER} = \$35{,}000\left[\frac{1}{(\$3{,}950{,}000 + \$3{,}873.150)/2}\right]$$

$$= .009 \text{ or } .09\%$$

An expense ratio of .09% is reasonable since the range is .4 to 1.5%. Of course, the higher the expense ratio, the more it takes away from the return on investment.

The Total Investment Expense (TIE)

The **total investment expense** is an expansion of the expense ratio. It adds to it the load expenses as they are adjusted to the holding period.

$$\boxed{\text{TIE} = \frac{1}{n}(L_f + L_B) + \text{ER}}$$

Example 3.3.2 Let's suppose that the front-end load of a mutual fund was 4% and the back-end load was 3.5% and the fund was held for 3 years. The total investment expense would be

$$\text{TIE} = \tfrac{1}{3}(.04 + .035) + .009$$

$$= .034 \text{ or } 3.4\%$$

The Reward-to-Variability Ratio (RVR)

The **reward – variability ratio** was developed by W. F. Sharpe in 1966 (*Journal of Business*, January, pp. 119 – 138). It assesses the mutual fund beyond the risk-free return for every unit of total risk that may face the fund. It compares the fund return to the risk-free return of 91-day U.S. Treasury bills and assesses the difference between the rates against the fund's standard deviation of returns.

$$\boxed{\text{RVR} = \frac{1}{\sigma_j}(R_{jt} - R_{ft})}$$

where R_{jt} is the return on the j th fund for time t and R_{ft} is the return on a risk-free asset, usually Treasury bills, and σ_j is the standard deviation of return on the j th fund.

Example 3.3.3 Suppose that at a time when a mutual fund return was 9%, the risk-free return was 5%, and the standard deviation of returns in the same time was 18.53. The reward-to-variability ratio would be

$$\begin{aligned}\text{RVR} &= \frac{1}{.1853}(.09 - .05) \\ &= 21.6\%\end{aligned}$$

which means that the fund is providing 21.6% return beyond the risk-free rate.

Note that if the fund rate of return has to be calculated, we would calculate it using the following:

$$\boxed{R = \frac{S}{\text{IF}}(.6.P + D + G)}$$

where R is the rate of return on mutual funds, S is the number of shares owned, IF is the amount invested in funds, $.6.P$ is the change in the price of the fund, D is the dividend received per share, and G is the capital gain per share.

Example 3.3.4 Suppose that we invested \$2,500 in purchasing 200 shares of a mutual fund at \$12.50 a share and after a year this price increased to \$13.70 and the company gave out 30 cents in dividends and 45 cents in capital gain. What would the rate of return be?

$$\begin{aligned}R &= \frac{S}{\text{IF}}(\Delta P + D + G) \\ &= \frac{200}{\$2{,}500}[(13.70 - 12.50) + .30 + .45] \\ &= 15.65\%\end{aligned}$$

Also, if the standard deviation of returns has to be calculated, we would calculate it by the normal statistical formula of the standard deviation:

$$\sigma = \sqrt{\frac{\sum(xt - \bar{x})2}{n - 1}}$$

where x_t is the return for period t, x is the mean return, and n is the number of periods. Standard deviation is often used to estimate the extent of risk in surrounding the flow of fund returns. A large standard deviation usually indicates a higher level of risk.

Example 3.3.5 Suppose that we track down the 7-year returns of a specific mutual fund and find them to be as follows:

$$5, \quad 5\tfrac{3}{4}, \quad 6\tfrac{1}{2}, \quad 6, \quad 7\tfrac{1}{3}, \quad 8, \quad 9\tfrac{1}{2}$$

We can calculate the standard deviation by first finding the mean, which would be 6.87, and then arranging the data we need in Table E3.3.5.

$$\sigma = \sqrt{\frac{14.06}{7-1}}$$

$$= \sqrt{2.34}$$

$$= 1.53$$

This is a relatively small standard deviation, indicating a lower level of risk. Similarly, we can follow other regular statistical measures when we need their aid for analysis, such as in the case of testing the degree of diversification of funds in an investor's portfolio. We can see how these funds are tied to each other and to what degree by employing a correlation index such as Pearson's

TABLE E3.3.5

t	x_t	$\bar{x}$	$x_t - \bar{x}$	$(x_t - \bar{x})^2$
1	5	6.78	−1.87	3.5
2	5.75	6.78	−1.12	1.25
3	6.5	6.78	−.37	.137
4	6	6.78	−.87	.757
5	7.34	6.78	.47	.221
6	8	6.78	1.13	1.28
7	9.5	6.78	2.63	6.92
				$\sum(x_t - \bar{x})^2$ 14.06

correlation coefficient (r_p):

$$r_p = \frac{1}{\sigma i\, \sigma j}\left[\frac{\sum(Ri - \overline{Ri})\,(Rj - \overline{Rj})}{n-1}\right]$$

where σ_i and σ_j are the standard deviations of returns for funds i and j, R_i and R_j are the series of returns of funds i and j, $\overline{R}_i$ and $\overline{R}_j$ are the means of those returns, and n is the number of return periods or observations.

Example 3.3.6 Suppose that we want to test how diversified Jim's investment portfolio of mutual funds is by looking at only two funds and observing their returns in the last six years (Table E3.3.6).

$$\sigma i = \sqrt{\frac{\sum (Ri - \overline{Ri})2}{n-1}}$$

$$= \sqrt{\frac{18.05}{6-1}}$$

$$= 1.9$$

$$= \sqrt{\frac{26.78}{6-1}}$$

$$= 2.31$$

$$r_p = \frac{1}{\sigma i\, \sigma j}\left[\frac{\sum(Ri - \overline{Ri})\,(Rj - \overline{Rj})}{n-1}\right]$$

$$= \frac{1}{1.9\,(2.31)}\left(\frac{-20.26}{6-1}\right)$$

$$= .923 \quad \text{or} \quad -92.3\,\%$$

This very high and negative correlation means that the funds in the portfolio are highly diversified. They would move in a way strongly opposite to each other,

TABLE E3.3.6

n	R_i	$\overline{R}_i$	$R_i - \overline{R}_i$	$(R_i - \overline{R}_i)^2$	R_j	$\overline{R}_j$	$R_j - \overline{R}_j$	$(R_j - \overline{R}_j)^2$	$(R_i - \overline{R}_i)(R_j - \overline{R}_j)$
1	5	7.04	−2.04	4.16	12	8.29	3.71	13.76	−7.57
2	5.5	7.04	−1.54	2.37	9	8.29	.71	.50	−1.09
3	6.25	7.04	−.79	.624	8.25	8.29	−.04	.0016	.032
4	7	7.04	−.04	.0016	8.5	8.29	.21	.044	−.0084
5	8.5	7.04	1.46	2.13	7	8.29	−1.29	1.66	−1.88
6	10	7.04	2.96	8.76	5	8.29	−3.29	10.82	−9.74
				18.05				26.78	−20.26

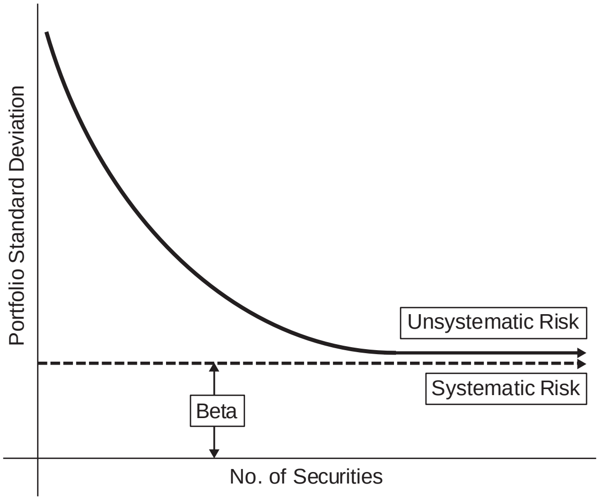

FIGURE 3.1

which is the aim of diversification. While one fund does not do well, the other does extremely well, and a nice balance is achieved.

Treynor's Index

Treynor's index is named after J. L. Treynor (*Harvard Business Review* , January – February 1965, pp. 63 – 75). It is similar to the share's reward-to-variability ratio (RVR) except that the difference between the rate of return generated by the mutual fund and the risk-free rate of U.S. Treasury bills is divided by beta (β_j) instead of σ_j where beta is a coefficient representing the estimated systematic risk that the fund in question may face.

$$\boxed{\text{TI} = \frac{1}{\beta_j}(R_{jt} - R_{ft})}$$

Example 3.3.7 If the coefficient estimated for the market systematic risk (β) at the time of calculating the previous rate of return is estimated at 1.07, the Treynor index would be

$$\begin{aligned} \text{TI} &= \frac{1}{1.07}(.09 - .05) \\ &= 3.7\% \end{aligned}$$

which says that this fund is delivering a 3.7% return beyond the risk-free return for every level of the systematic risk that the fund was expected to return.

1.3.3 THE EFFECT OF SYSTEMATIC RISK (β)

Systematic risk (β) is also called **market risk** or **undiversified risk**. It refers to the unavoidable type and level of risk that is inherent in the nature of the free market and tied inextricably to its fluctuations. In the context of a mutual funds portfolio, this type of risk cannot be eliminated or reduced by the usual remedy of diversifications of funds. It is associated with the fact that there are other economy-wide perils that continue to threaten all business performance, and it is the reason that stocks, for example, tend to move together. It is an inevitability that investors have to deal with. Figure 3.1 shows how the level of systematic risk is determined independently with regard to the unsystematic or specifically unique level of risk to which securities are individually exposed.

Systematic risk for a specified fund return such as R_i is measured by beta (β_i), which is obtained by dividing the covariance between R_i and market return (R_m) by the variance of market returns:

$$\beta_i = \frac{\mathrm{Cov}(R_i,\, R_m)}{\sigma_m^2}$$

Keeping in mind that for a risk-free return (R_f), beta would be zero because its covariance with the market return is zero, and for the collective market returns, beta would be equal to 1 because the covariance of market return with itself is the variance of itself:

$$\beta_m = \frac{\mathrm{Cov}(R_m,\, R_m)}{\sigma_m^2} = \frac{\sigma_m^2}{\sigma_m^2} = 1$$

As for many individual funds in a portfolio, beta would be the weighted average of all individual betas of the returns of the portfolio funds:

$$B_p = \sum_{j=1}^{k} Bj \,\square j \qquad j = 1,2, \ldots,k$$

where β_p is the beta for the portfolio, β_j is the individual beta for each fund in the portfolio, and w_j is the weight of each fund in the portfolio, which is simply the proportion of the fund market value to the entire value of the portfolio.

Example 3.4.1 Suppose that we have a portfolio of five funds, the market values of which are \$250,000, \$370,000, \$588,000, \$610,000, and \$833,000, with estimated betas of .8, .75, 1.2, 1.35, and .97 (see Table E3.4.1). Calculate the portfolio beta, β_p.

First, we obtain the market value of the portfolio MV_p and then we calculate the funds weights (w_j) as the individual fund percentages of the entire portfolio:

$$\mathrm{MV}_p = \sum_{j=1}^{5} MVj = \$250.000 + \$370{,}000 + \$588{,}000 + \$610{,}000 + \$833{,}000$$

$$= \$2{,}651{,}000$$

TABLE E3.4.1

Fund	MV_j	W_j	B_j	$B_j\ W_j$
1	250,000	.094	.8	.075
2	370,000	.139	.75	.104
3	588,000	.222	1.2	.2666
4	610,000	.231	1.35	.312
5	833,000	.314	.97	.305
Portfolio	2,651,000			$\beta_p = 1.06$

$$W_1 = \frac{MV_1}{MV_p} = \frac{\$370,000}{\$2,651,000} = 9.4\%$$

$$W_2 = \frac{MV_2}{MV_p} = \frac{\$250,000}{\$2,651,000} = 13.9\%$$

$$W_3 = \frac{MV_3}{MV_p} = \frac{\$588,000}{\$2,651,000} = 22.2\%$$

$$W_4 = \frac{MV_4}{MV_p} = \frac{\$610,000}{\$2,651,000} = 23.1\%$$

$$W_5 = \frac{MV_5}{MV_p} = \frac{\$833,000}{\$2,651,000} = 31.4\%$$

The Abnormal Performance Alpha (α_{jt})

The **abnormal performance alpha** measure is also called **Jensen's index** after M. Jensen (*Journal of Finance*, May 1968, pp. 389 – 416), who used it to test the abnormality in mutual fund performance by calculating the coefficient alpha (α), which compares two rate differences:

1. The difference between the returns of the performing fund and a risk-free asset such as U.S. Treasury bills.
2. The difference between the market (R_{mt}) return and the risk-free asset return, adjusted for systematic risk.

$$\boxed{\alpha_{it} = (R_{jt} - R_{ft}) - [\beta_j(R_{mt} - R_{ft})]}$$

Jensen explained that a positive alpha means that after adjusting for risk and movements in the market index, the abnormal performance of a portfolio stays on, and that the fund is able to handle its own expenses. A negative alpha suggests that the fund is not able to forecast future security prices well enough to cover expenses. Jensen measured primarily net costs such as research cost, management fees, and brokerage commissions.

Example 3.4.2 If we keep the same data for the rate we used before and assume that the market rate is 10%, we can calculate alpha as

$$\alpha_{it} = (.09 - .05) - [1.07(.10 - .05)]$$
$$= 1.35\%$$

1.3.4 DOLLAR-COST AVERAGING

One of the simplest and most popular techniques in the mutual fund trust is **dollar-cost averaging**, which minimizes the cost and increases the return over the long run. It is used basically to maintain a regular periodic investment that over time would automatically purchase more of low-priced shares and fewer high-priced shares, striking a natural balance. The result is that the average cost per share will always be less than the average price. There are two ways to achieve regularity in investment:

1. Buying the same number of shares regardless of the cost per share.
2. Buying for the same dollar amount regardless of the cost per share.

Either way should result in an average cost that is less than the average price per share.

Example 3.5.1 Suppose that we invest $250 a month to buy shares in a specific fund throughout its price changes over the next six months: July, 14.25; August, 13.35; September, 12.20; October, 11.92; November, 13.10; and December, 12.50 (Table E3.5.1). Calculate the average cost and compare it to the average price.

$$\text{average cost} = \frac{\sum_{i=1}^{6} Ii}{\sum_{i=1}^{6} Si} = \frac{1{,}500}{116.82} = 12.84$$

$$\text{average price} = \frac{\sum Pi}{n} = \frac{77.32}{6} = 12.89$$

TABLE E3.5.1

Month, n	Share Price, P_i ($)	Amount Invested, I_i ($)	No. Shares, S_i
July	14.25	250	17.54
August	13.35	250	18.73
September	12.20	250	20.50
October	11.92	250	20.97
November	13.10	250	19.08
December	12.50	250	20.00
Total	77.32	1,500	116.82

UNIT II

Mathematics of Investment

Options

As a form of investment, the organized options market has been advancing rapidly in the last four decades. It began in 1973 when the Chicago Board of Options Exchange (CBOE) ushered in the era of trading in standardized option contracts. Since then it has became a significant part of the investment scene, has attracted its own zealous investors, and has introduced its own culture and language, which we need to explore. **Options** are financial instruments that act like a standardized contract and provide their holders with the opportunity and right, but not the obligation, to purchase or sell a certain asset, at a stated price, on or before a short-term expiration date, usually set as within a year.

There are three basic forms of options: rights, warrants, and calls and puts. We focus on the most common: call options and put options. A **call option** gives its holder the right to purchase a specific number of shares (usually, 100 shares of common stock) at a price called a **strike** or **exercise price**, on or prior to a specific expiration date. The strike price is often set at or near the prevailing market price of a stock at the time the option is issued. A **put option** is similar in every way to a call option except that it gives the right to sell instead of to purchase. Although the most common underlying assets for options are common stocks, they can also be based on stock indices, foreign currencies, debt instruments, and commodities. Those underlying assets are why options are considered derivatives: because the option value depends on the value of its basis asset. This is also what justifies the existence of options. It is because investors expect the market price of the underlying asset to rise high enough to cover the cost of the option and still leave room to make a profit.

The option trading cycle begins when an option contract is generated by an **option writer**, who would be paid a premium by the asset owner to create such an option, and grants its selling and buying rights that can be exercised during the set life of that option. An **option holder** can monitor his or her action based on the market prices of the underlying assets and the level of risk involved. Generally, the option holder may follow one of the following actions:

1. *To exercise the option:* to exercise the right to buy or sell an option. When the market price of the underlying asset, such as a common stock, is higher than the strike price of the option, a holder of a call option would exercise his or her right to buy a call and make a profit. Similarly, when the market

price is lower than the strike price of the put option, the put holder would exercise his or her right to sell and make a profit. This is the case called *in-the-money*.

$$\text{In-the-money:} \quad \begin{array}{l} MP_c > SP \\ MP_p < SP \end{array}$$

MP_c is the market price of a call, MP_p is the market price of a put, and SP is the strike price.

2. *Not to exercise the option:* when the investor does not see any opportunity to make a profit. It is usually the case when the market price of the underlying asset is either equal to or lower than the strike price of a call and either equal or higher for a put. This case is called *out-of-the-money* when prices are different and *at-the-money* when prices are equal.

   Out-of-the-money:

$$MP_c < SP$$

$$MP_p > SP$$

   At-the-money:

$$MP_c = SP$$

$$MP_p = SP$$

3. *To let the option expire:* when investors keep waiting for the prices of the underlying assets to change in their favor so that they can exercise the option. Sometimes, no change in price of this sort occurs and investors cannot take any action until time runs out, the expiration date arrives, and their options are deemed worthless. In this case, investors suffer a loss. There are strategies used by financial institutions to protect investors against this loss and other types of loss in the investment market. One very common strategy, **hedging**, is an action taken with one security to protect another security against risks such as buying on one side and selling on the other. In option investment there are three common types of strategies: spreads, straddles, and a combination of both. A **spread** is a simultaneous purchase and sale of calls and puts on the same underlying asset, such as stock, which are written with either different strike prices, different expiration dates or both. Three spreads can be identified:
   a. A **horizontal spread** is characterized by buying and selling an option with an identical strike price but different expiration dates.
   b. A **vertical spread** involves the purchase and sale of an option with identical expiration dates but a different strike price.
   c. A **butterfly** represents many options purchased at different strike prices.

A **straddle** involves a combination of the same number of calls and puts purchased simultaneously at an identical strike price for the same expiration date.

2.1 DYNAMICS OF MAKING PROFITS WITH OPTIONS

Investors make profits by buying and selling calls and puts and through the differences between the market price of the underlying assets and the strike price of those options exercised within the proper timing.

Buying and Selling Calls

An investor who buys calls has the right to purchase 100 shares of the underlying asset (let's say stock here) per call at a strike price that would stay valid for a certain maturity time. This investor would watch the fluctuations of the market price of that stock and hope that it would rise above the strike price that she paid so that she can sell and profit from the difference between the two prices, all before the expiration day, which is the last day of the maturity. This type of profit can be substantial and achieved in a short period of time, but comes with a high risk. The basic risk is that the maturity time will expire without the investor being able to sell at a profit. It is very possible that the market price of such stocks would not rise within the time frame of maturity, rendering those calls worthless. Some investors try to sell at attractive discounts before the expiration date, to recover at least a portion of their investment.

There are two common ways of selling a call: the risky and the conservative. In the risky way, called an **uncovered sale**, the seller grants to the buyer the contractual right that 100 shares will be delivered at the strike price no matter how high the market value of the stock. Suppose that a seller grants such a right to a buyer to deliver 100 shares of a stock at a strike price of $30, and suppose by the expiration date the stock market price rises to $59. The seller would suffer a significant loss: $2,900. The less risky type of call sale, the **covered sale**, involves selling shares when the seller already owns those shares instead of having to buy them, even when the market value is substantially high. It is most likely that they were purchased at a lower price, so that this sale is significantly less risky than an uncovered sale of calls.

Buying and Selling Puts

In contract to a call investor, the buyer of a put waits for the market price of a stock to fall below the strike price so that she can make a profit by buying more cheaply. But just like calls, puts can easily lose their value if a drop in the market value of the underlying stock does not occur by the expiration date. In selling puts, there are no covered and uncovered sales as in the case of selling calls. The seller would wish for the market price of the underlying stock to rise above the strike price so that he can make a profit when exercising his option by selling at a higher value than that at which he purchased the put.

2.1.1 INTRINSIC VALUE OF CALLS AND PUTS

The intrinsic value of a call option to the buyer is what he would gain through the difference between the market price of the underlying stock and the strike price. He would gain only when the market price of the underlying stock exceeds the strike price when he chooses to exercise his option right to buy at the strike price and sell at the market price. However, he would not lose anything if the market price of the stock decreases, because he has the choice not to exercise his option but wait for a better opportunity. The intrinsic value for the call writer is what he loses when the market price of the underlying stock increases at the time he issued the option at a strike price less than the current market price of the underlying stock.

The value in both cases, buying a call and writing a call, is determined dollar by dollar by how much the market price of a stock increases over the strike price, but it is determined inversely for both of them (i.e., what the buyer gains, the writer loses). Mathematically, the intrinsic value of a call to a call buyer (IVC_B) and for a call writer (IVC_W) can be expressed by

$$\begin{aligned}\mathrm{IVC}_B &= \max[(\mathrm{MP} - \mathrm{SP}), 0]\\ \mathrm{IVC}_W &= \min[(\mathrm{SP} - \mathrm{MP}), 0]\end{aligned}$$

where MP is the market price of the stock and SP is the strike price for the call option.

Example 4.2.1 Suppose that a call option is written for a strike price of \$25 per share of a certain stock, and suppose that after some time, the market price for that stock goes up to \$37. What is the intrinsic value for the buyer and the writer?

$$\begin{aligned}\mathrm{IVC}_B &= \max[(\mathrm{MP} - \mathrm{SP}), 0]\\ &= \max[(\$37 - \$25), 0]\\ &= \max[\$12, 0]\\ &= \$12\\ \mathrm{IVC}_W &= \min[(\mathrm{SP} - \mathrm{MP}), 0]\\ &= \min[(\$25 - \$37), 0]\\ &= \min[-\$12, 0]\\ &= -\$12\end{aligned}$$

So the value to the buyer is \$12 and to the writer is –\$12, which means that the writer loses as much as the buyer gains because the writer had to stay on her contractual obligation to deliver the option at the strike price of \$25 even if she had to buy the stock at \$37 in order to make it available to the buyer.

In Figure E4.2.1 we see that everything was flat before the \$25 price, but after the market price increased to \$37, each dollar of that increase raised the potential

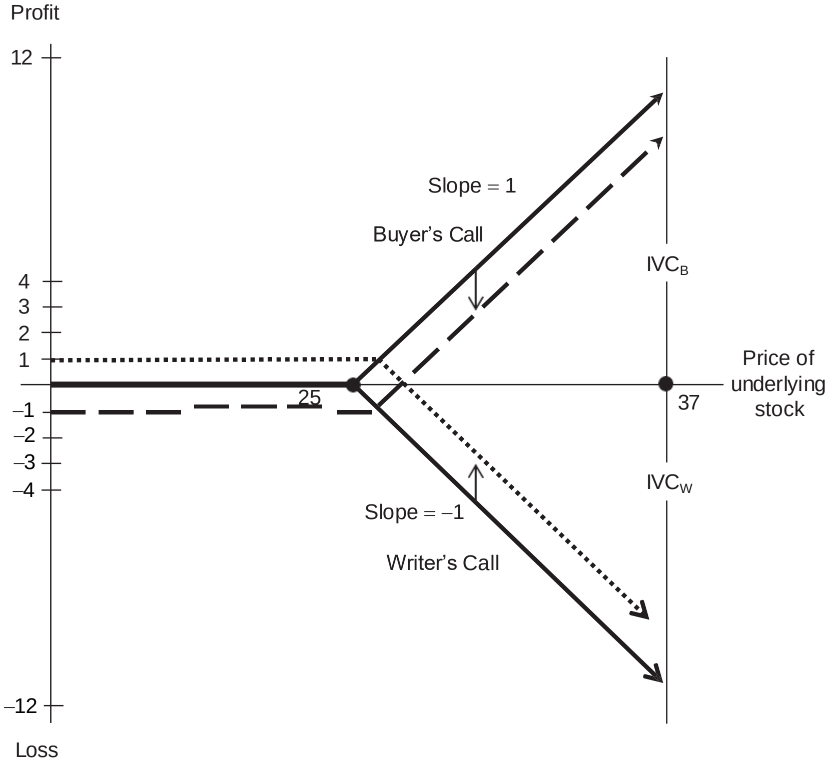

FIGURE E4.2.1

profit by the same dollar amount, which is why the slope is 1 and the line is at 45°. The increase was $12 and the gain was also $12. It is exactly the opposite for the writer, whose curve dropped by each dollar increase in the stock price and ultimately became a mirror image of the buyer's call curve. The entire loss was also $12.

Note that in the calculation we did not account for the fee paid to the call writer, called the **option premium**. In the figure, the fee is shown to reduce the gain for the buyer by shifting the entire curve down to the dashed line. At the same time, this fee is received by the writer, reducing her loss by shifting her curve up to the dotted line.

From the perspective that puts are the opposite of calls, we can see that the intrinsic value equation for a put would be the same as the call value equation except in switching the order of the prices. Therefore, we can write the intrinsic value of a put to a buyer (IVP_B) and to a put writer (IVP_W) as

$$\text{IVP}_B = \max[(\text{SP} - \text{MP}), 0]$$
$$\text{IVP}_W = \min[(\text{MP} - \text{SP}), 0]$$

In this case, the put buyer seeks a drop in the market price of the underlying stock so that he can make a potential profit through buying cheap. The put writer, on the other hand, would make a loss by delivering an option at a strike price higher than what the market sells. Again, in both cases, what the buyer gains, the writer loses, and both gain and loss are dollar for dollar the same as the drop in the market price of the stock.

Example 4.2.2 An investor has a put option with a strike price of \$70, but the market price of its underlying stock goes down to \$65 (see Figure E4.2.2). What is the intrinsic value of the put for this buyer and for the writer?

$$\begin{aligned} \mathrm{IVP}_B &= \max[(\mathrm{SP} - \mathrm{MP}), 0] \\ &= \max[(\$70 - \$65), 0] \\ &= \max(\$5, 0) \\ &= 5 \end{aligned}$$

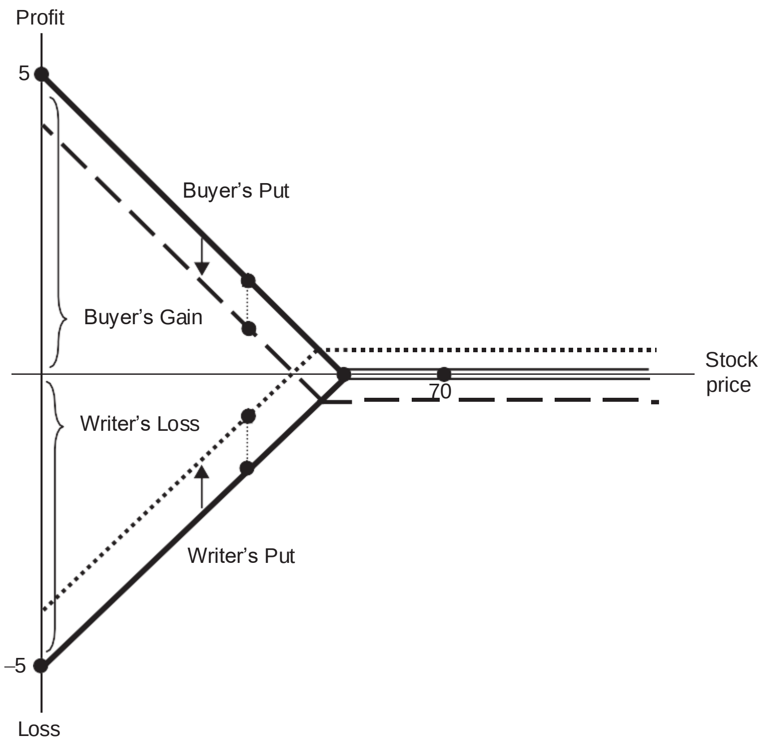

FIGURE E4.2.2

$$
\begin{aligned}
\mathrm{IVP}_W &= \min[(\mathrm{MP} - \mathrm{SP}), 0] \\
&= \min[(\$65 - \$70), 0] \\
&= \min(-\$5 - 0) \\
&= -\$5
\end{aligned}
$$

Similar to the call case, what the buyer gains, which is $5 per share, the writer loses, and both gain and loss are dollar for dollar as much as the drop in the market price of the stock. The reason behind the put buyer's gain is that he pays $70 as a contracted strike price, but the writer is committed to deliver a $65 per share, although she has to carry the loss of $5 per share.

As expected, the graphic presentation of the put payoff would be very similar to the call payoff except that it is switched to the left side. The reason for the switch is that the entire sequence follows a drop in the market price of the stock, not an increase, which warrants the movement from right to left. Again, the actual buyer's and writer's curves are the dashed line for the buyer, which is shifted down since it reflects the reduction caused by paying the option premium. The dotted line represents the actual writer's curve, which is shifted up to reflect the gain in receiving the option premium.

2.1.2 TIME VALUE OF CALLS AND PUTS

The time value of an option, a call (TV_c) or a put (TV_p), is the difference between the option price (OP) and the intrinsic value of the option, whether it is for the call (IVC) or for the put (IVP). It is also defined as the portion of the premium above any in-the-money premium.

$$\mathrm{TV}_c = \mathrm{OP} - \mathrm{IVC}$$

and

$$\mathrm{TV}_p = \mathrm{OP} - \mathrm{IVP}$$

Example 4.3.1 A call option is currently worth $600 for a 100 share, and its strike price is $40. Find the call time value for its buyer if the market price of the stock rises to $45.

$$
\begin{aligned}
\mathrm{IVC}_B &= \max[(\mathrm{MP} - \mathrm{SP}), 0] \\
&= \max[(\$45 - \$40), 0] \\
&= \max[\$5, 0] \\
&= \$5
\end{aligned}
$$

$$\mathrm{OP} = \frac{\$600}{100} = \$6 \text{ per share}$$

$$
\begin{aligned}
\mathrm{TV}_C &= \mathrm{OP} - \mathrm{IVC} \\
&= \$6 - \$5 = \$1.00
\end{aligned}
$$

Example 4.3.2 What is the intrinsic value of an option if it costs \$500 (for 100 shares), and its time value is \$2?

$$\text{OP} = \frac{\$500}{100} = \$5$$

$$\begin{aligned} \text{IV} &= \text{OP} - \text{TV} \\ &= \$5 - \$2 = \$3 \end{aligned}$$

Example 4.3.3 If we are told that the option in Example 4.3.2 was a call and the market price of its underlying stock was \$75, what would the strike price be that has been granted to the buyer?

The intrinsic value of a call for the buyer is ultimately equal to the difference between the market price of the stock and the strike price of the call:

$$\begin{aligned} \text{IVC}_B &= \text{MP} - \text{SP} \\ \text{SP} &= \text{MP} - \text{IVC}_B \\ &= \$75 - \$3 = \$72 \end{aligned}$$

Note that the time value would be equal to the premium or the option price if the option is either at-the-money or out-of-the-money. In the case of at-the-money, the market price of the stock and the strike price would be equal and the intrinsic value would be zero. The time value would be the premium minus zero, which would end at the equality between the time value and the premium. In the case of out-of-the-money, the difference between the market price of the stock and the strike price would be negative, lending to a consideration of zero as the intrinsic value to signify that such an option would have no intrinsic value. Here, too, the time value would be the result of taking away zero from the premium, which means keeping the premium value as it is. Table 4.1 shows how this works.

As an example, when we look at calls 3 and 10 and puts 1 and 6 (marked by arrows), we observe that the time value equals the premium OP $=$ TV (4's; 10's; 1's; and 3's) simply because the intrinsic values were zero. Also, we can observe that the intrinsic value cannot exist ($= 0$) if the market price is lower than the strike price in the case of calls or higher than the strike price in the case of puts (cases marked by stars). Therefore, zero is assigned to the intrinsic value in these cases. Naturally, the intrinsic value would be zero when the market price equals the strike price (cases marked by x).

2.1.3 THE DELTA RATIO

Delta (D) describes the relationship between the change in the market price of the underlying stock ($.6.$MP) and the change in option price or the premium

(.6.OP). It is an index that tells investors about the dynamics of their profits and losses out of option trading. Mathematically, it is a ratio of the changes in these two prices:

$$\boxed{D = \frac{\Delta \text{OP}}{\Delta \text{MP}}}$$

If delta is equal to 1, it means that the option price follows the market price dollar for dollar. If it is more than 1, the option price would respond faster to the change in market price, and if it is less than 1, we know that the option price would lag behind in its response to the change in market price. Table 4.1 shows that cases of calls 1 and puts 2, 3, and 5 are where the option price and market price go together. Calls 4, 5, and 8 and puts 9 and 10 are where the option prices were ahead of the market prices, and finally, calls 6, 9, and 10 and puts 8 are where the option prices stayed behind in their response to the market change.

TABLE 4.1

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
			IV		TV			*D*
Option	MP	SP	[(1) − (2)]	OP	[(4) − (3)]	.6.OP	.6.MP	[(6) ÷ (7)]
Calls								
x 1	17	17	0	2	2			
2	19	18	1	4	3	2	2	1
→ 3	20	20	**0**	**4**	**4**	0	1	0
4	22	20	2	7	5	3	2	1.5
* 5	21	23	0	5	5	–2	–1	2
6	23	22	1	6	5	1	2	.5
* 7	24	25	0	6	6	0	1	0
8	28	25	3	11	8	5	4	1.25
* 9	25	28	0	9	9	−2	−3	.66
→ 10	27	27	**0**	**10**	**10**	1	2	.5
	(1)	(2)	(3) = (2) − (1)	(4)	(5) = (4) − (3)	(6)	(7)	(6) ÷ (7)
Puts								
→ 1	17	15	**0**	**1**	**1**			
2	19	20	1	3	2	2	2	1
x 3	20	20	0	4	4	1	1	1
* 4	22	20	0	4	4	0	2	0
5	21	23	2	3	1	–1	–1	1
→ 6	23	22	**0**	**3**	**3**	0	2	0
* 7	24	23	0	3	3	0	1	0
* 8	28	25	0	6	6	3	4	.75
x 9	25	25	0	2	2	–4	–3	1.3
10	27	29	2	5	3	3	2	1.5

2.1.4 DETERMINANTS OF OPTION VALUE

Five major factors determine the market value of an option, especially a call option, since it is the most common and popular option in the exchange market. The following equation describes the value of a call option (VC) as a function of those factors:

$$\mathrm{VC} = f[\mathrm{MP},\ \mathrm{SP},\ T, r_f, \sigma^2_{mp}]$$

$$\frac{\partial \mathrm{VC}}{\partial \mathrm{MP}} > 0 \qquad \frac{\partial \mathrm{VC}}{\partial T} > 0 \qquad \frac{\partial \mathrm{VC}}{\partial r_f} > 0 \qquad \frac{\partial \mathrm{VC}}{\partial \sigma^2_{\mathrm{mp}}} > 0$$

but

$$\frac{\partial \mathrm{VC}}{\partial \mathrm{SP}} < 0$$

where MP is the market price of the underlying stock that would affect the call value positively. The higher the market price of stock, the greater the call value, *ceteris paribus*; SP is the strike price of the call option, which affects the value of option negatively. The lower the strike price, the greater the call value, *ceteris paribus*. T is the length of maturity time, r_f is the risk-free rate, and σ_{mp}^2 is the variance in the market price of the underlying stock. All of these last three factors affect the value of the call option in a positive manner.

Figure 4.1 shows two hypothetical distributions of the market price of the underlying stock. Distribution 2 has a higher variance, and therefore the price movement is harder to predict and more volatile than distribution 1. Both distributions are assumed to have the same expected stock price and same strike price. Since the call option is characterized by being a contingent claim (i.e., the

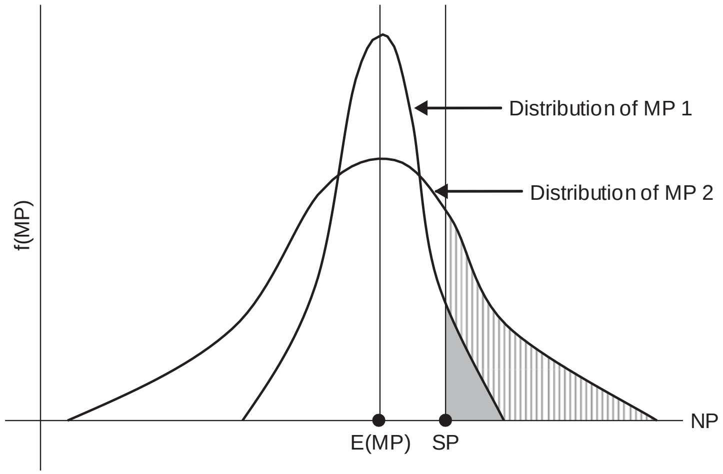

FIGURE 4.1

call holder would make a profit only when the stock price is greater than the strike price), the second distribution with the larger variance would offer much higher probability for the market price to exceed the strike price compared with the limited probability offered by the first distribution.

Consequently, we can conclude that the same set of determinants also affects the value of the put (VP):

$$\mathrm{VP} = f[\mathrm{MP},\ \mathrm{SP}, T, r_f, \sigma_{mp}{}^2]$$

$$\frac{\partial \mathrm{VP}}{\partial \mathrm{MP}} < 0 \qquad \frac{\partial \mathrm{VP}}{\partial r_f} < 0$$

$$\frac{\partial \mathrm{VP}}{\partial \mathrm{SP}} > 0 \qquad \frac{\partial \mathrm{VP}}{\partial T} > 0 \qquad \frac{\partial \mathrm{VP}}{\partial \sigma^2_{\mathrm{mp}}} > 0$$

But the dynamics of the factor changes are different. The market price of the stock and the risk-free rate affect the value of puts negatively, but the rest of the factors — the strike price, the time of maturity, and the variance of the market price — affect the value of puts positively.

2.1.5 OPTION VALUATION

The 1973 study by F. Black and M. Scholes (*Journal of Political Economy* , **81**, 637 – 654) on option pricing became a classic reference in the option evaluation calculation. The call option value (VC) is determined by

$$\boxed{\mathrm{VC} = \mathrm{MP}[N(d_1)] - (e)^{-rT} \cdot \mathrm{SP}[N(d_2)]}$$

where MP is the market price of the underlying stock, SP is the strike price of the call, r is the risk-free rate, T is the length of maturity time, and $N(d)$ is the cumulative normal probability density function. VC stands for the probability that a normally distributed random variable will be less than or equal to the area d, where

$$\mathrm{d1} = \frac{In\ (MP\ /\ SP) + (r + \sigma 2/2)}{\sigma \sqrt{\mathrm{T}}}$$

and

$$\mathrm{d2} = \mathrm{d1} - \sigma \sqrt{T}$$

where σ is the standard deviation per period of the continuously compounded rate of return on stock. As for the value of a put (VP), it can be obtained using the calculated value of the call (VC), the current value of the strike price CV(SP), and the market price of the stock:

$$\text{VP} = [\text{VC} + \text{CV}(\text{SP})] - \text{MP}$$

$$\boxed{\text{VP} = [\text{VC} + \text{SP}(e)^{-rT}] - \text{MP}}$$

Although these formulas may sound tedious in application and require obtaining two table values, $N(d_1)$ and $N(d_2)$, Black and Scholes' formulas were simplified for the hand calculation even though computerized programs can handle much more complex values in a split second. The following is the simplified way to get the call and put values:

$$\boxed{\text{VC} = \text{MP}(\text{PSP value})}$$

where the value of a call is a certain percentage of the market price of stock. This percentage is determined by a table value called the **percentage of share price** (PSP). Look at Table 9 in the Appendix. PSP is obtained according to two calculated values:

1. *Vertical value:* calculated as the product of the standard deviation (σ) and the square root of maturity $\sqrt{T}$:

$$\sigma\sqrt{T}$$

2. *Horizontal value:* calculated by dividing the market price of stock by the current value of the strike price using the risk-free rate and the actual maturity time in annual format.

$$\boxed{\frac{\text{MP}}{\text{CV(SP)}} = \frac{\text{MP}}{\text{SP}/(1 + r_f)^T}}$$

Example 4.6.1 Calculate the value of a call option with a strike price of \$90 and a maturity of 9 months. Given that the current stock price is \$108.50, the risk-free rate is 5% and the standard deviation of the rate of return on stock is .75.

First we have to look up the PSP value in Table 9 in the Appendix, but we need to calculate the vertical and horizontal values first:

- Vertical value: $\sigma\sqrt{T} = (.75)\ \sqrt{.75}$ considering the nine-month maturity as three-fourths of the year

$$\sigma\sqrt{T} = .65$$

- Horizontal value

$$\frac{\text{MP}}{\text{SP}/(1+r_f)^T} = \frac{\$108.50}{\$90/(1+.05)^{.75}} = \$1.25$$

Next, we look at the value in Table 9 corresponding to the vertical value of .65 and the horizontal value of $1.25. The value is 34.2 and that is the PSP value we need (it is in a percentage format).

$$\begin{aligned} \text{VC} &= \text{MP}(\text{PSP}) \\ &= \$108.50(.342) \\ &= \$37.11 \end{aligned}$$

Example 4.6.2 What would be the value of the put associated with the Example 1.1.1 call option?

$$\text{VP} = [\text{VC} + \text{CV}(\text{SP})] - \text{MP}$$

Recall that the current value of the strike price was the denominator of the horizontal value calculated above, which was

$$\begin{aligned} \text{CV(SP)} &= \frac{\text{SP}}{(1+r_f)^T} \\ &= \frac{\$90}{(1+.05)^{.75}} = \$86.77 \\ \text{VP} &= (\$37.11 + \$86.77) - \$108.50 \\ &= \$15.38 \end{aligned}$$

2.1.6 COMBINED INTRINSIC VALUES OF OPTIONS

One of the best known business strategies to diversify assets and protect against risks is **hedging**. It is used to combine options and create a mix of possibilities and returns through utilizing a variety of elements, such as different maturities, strike prices, and market prices. We have briefly described the types of option mixes in the early pages of this chapter. Below we calculate the combined intrinsic values of options in two forms of those combinations: a straddle and a butterfly spread.

Example 4.7.1 Calculate the intrinsic value of a combined option consisting of buying a call at a strike price of $55 and buying a put at the same strike price when the market price of the stock is $68. Calculate the value when the stock price goes down to $50.

$$\begin{aligned} \text{combined intrinsic value} &= \text{intrinsic value of call} + \text{intrinsic value of put} \\ &= \text{IVC}_B = \text{IVP}_B \end{aligned}$$

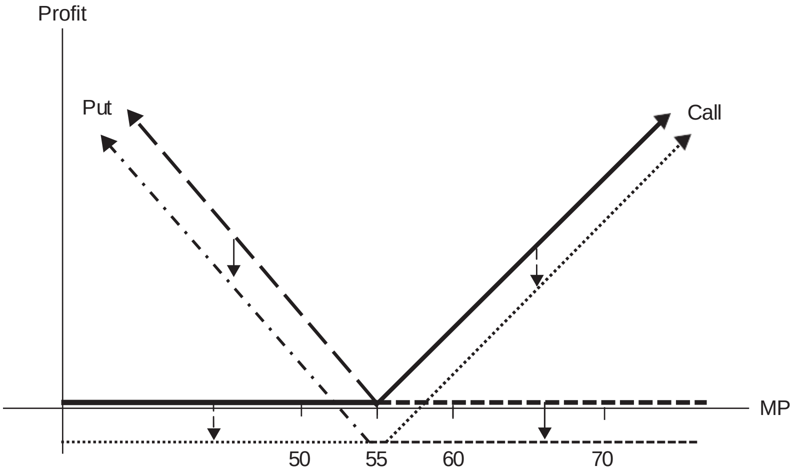

FIGURE E4.7.1 A straddle payoff.

$$
\begin{aligned}
1. \quad & = \max[(\text{MP} - \text{SP}), 0] + \max[(\text{SP} - \text{MP}), 0] \\
& = \max[(\$68 - \$55), 0] + \max[(\$55 - \$68), 0] \\
& = \max(\$13, 0) + \max(-\$13, 0) \\
& = \$13 \\
2. \quad & = \max[(\$50 - \$55), 0] + \max[(\$55 - \$50), 0] \\
& = \max(-\$5, 0) + \max(\$5, 0) \\
& = \$5
\end{aligned}
$$

This combination, called a **straddle**, is illustrated in Figure E4.7.1.

Example 4.7.2 Suppose that a small business investing in options buys two calls, whose strike prices are \$20 and \$30, and writes two puts, whose strike prices are \$25 and \$15. Calculate the intrinsic value of this combination when the stock price is \$40 and when it goes down to \$30.

When MP = \$40:

$$
\begin{aligned}
\text{IVC}_B^1 & = \max[(\text{MP} - \text{SP}), 0] \\
& = \max[(\$40 - \$20), 0] \\
& = \max(\$20, 0) = \$20
\end{aligned}
$$

$$\begin{aligned}
\text{IVC}_B^2 &= \max[(\$40 - \$30), 0] \\
&= \max(\$10, 0) = \$10 \\
\text{IVP}_W^1 &= \min[(\text{MP} - \text{SP}), 0] \\
&= \min[(\$40 - \$25), 0] \\
&= \min(\$15, 0) = \$0 \\
\text{IVP}_W^2 &= \min[(\$40 - \$15), 0] \\
&= \min(\$25, 0) = \$0
\end{aligned}$$

The combined intrinsic value = \$20 + \$10 + \$0 + \$0 = \$30.

When MP = \$30:

$$\begin{aligned}
\text{IVC}_B^1 &= \max[(\text{MP} - \text{SP}), 0] \\
&= \max[(\$30 - \$20), 0] \\
&= \max(\$10, 0) = \$10
\end{aligned}$$

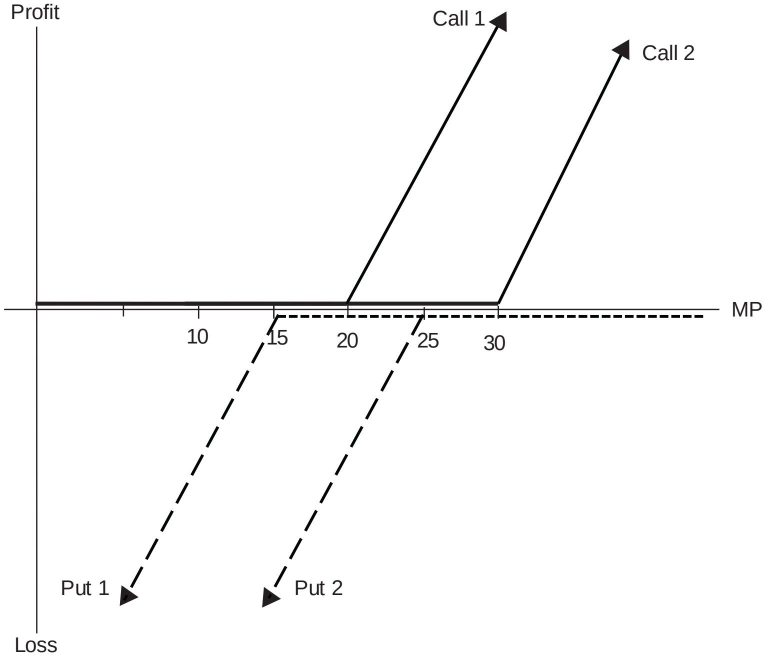

FIGURE E4.7.2 A butterfly spread.

$$\begin{aligned}
\mathrm{IVC}_B^2 &= \max[(\$30 - \$30), 0] \\
&= \max(\$0, 0) = \$0 \\
\mathrm{IVP}_W^1 &= \min[(\mathrm{MP} - \mathrm{SP}), 0] \\
&= \min[(\$30 - \$25), 0] \\
&= \min(\$5, 0) = \$0 \\
\mathrm{IVP}_W^2 &= \min[(\$30 - \$15), 0] \\
&= \min(\$15, 0) = \$0
\end{aligned}$$

The combined value = \$10 + \$0 + \$0 + \$0 = \$10.

This type of combination of options with different strike prices, called a **butterfly spread**, is illustrated in Figure E4.7.2.

Cost of Capital and Ratio Analysis

The **cost of capital** is a crucial concept in the context of financial decision making, especially in terms of being the accepted criteria by which a firm would decide whether an investment can or cannot potentially increase the firm's stock price. It is defined as:

1. The rate of return that the firm must earn on its investment to maintain a proper market value for its stock.
2. The rate of return that the investor must require to make its capital attractive for rewarding investment opportunities.

2.2 BEFORE- AND AFTER-TAX COST OF CAPITAL

If a firm uses a long-term debt such as selling bonds to finance its operation, a before-tax cost of debt can be calculated as

$$\boxed{CC_b = \frac{I + [(M - \mathrm{NP})/n]}{(\mathrm{NP} + M)/2}}$$

where CC_b is the cost of capital for bonds, I is the annual interest, M is the face value of bonds, NP is the net proceeds, which is the face value adjusted to the flotation cost, and n is the number of years to redemption.

Example 5.1.1 Suppose that a corporation is planning to collect a capital of \$5 million by selling its bonds of \$1,000, $8\frac{1}{2}$ % coupon rate. Given that the firm is selling at a discounts of \$30 per bond and that the flotation cost is 2% per bond, calculate the 20-year and the before-tax cost of capital.

$$\begin{aligned}
I &= \$1{,}000(.085) = 85 \\
B_d &= M - D \\
&= \$1{,}000 - \$30 = \$970 \\
\mathrm{NP} &= \$970 - (\$970 \times .02) = \$950.60
\end{aligned}$$

$$CC_b = \frac{I + [(M - \text{NP})/n]}{(\text{NP} + \text{N})/2}$$

$$= \frac{85 + [(\$1{,}000 - \$950.60)/20]}{(\$950.60 + \$1{,}000)/2}$$

$$= \frac{87.47}{\$975.30} = .09 \qquad \text{before-tax cost of capital is 9\%}$$

To get the after-tax cost of capital (CC_a), we use

$$CC_a = CC_b(1 - T)$$

where T is the corporate tax rate. Suppose that the corporate tax rate is 39%; then the after-tax cost of capital would be

$$CC_a = .09(1 - .39) = .055 \text{ or } 5.5\%$$

2.2.1 WEIGHTED-AVERAGE COST OF CAPITAL

A firms **capital structure** is the mix of debt and equity used to finance the firm's operation. The cost of many basic long-term sources of capital, such as stocks and bonds, have been detailed before. What remains is how these types of capital relate to the firm's capital structure and that is what the overall weighted-average cost of capital does. It is a method to determine the cohesiveness of the firm's capital structure by weighting the cost of each capital component based on its proportion as measured by the market value or book value.

$$CC_{wa} = \sum_{i=1}^{n} \square\square k_i$$

where CC_{wa} is the cost of capital weighted average, w_i is the proportion of any type of capital in the firm's capital structure, and k_i is the cost of any type of capital.

Example 5.2.1 The components of a corporation's capital structure and their individual costs are ontlined below and in Table E5.2.1.

1. Long-term debt takes 38% of capital structure and costs 5.59%.
2. Preferred stock represents 14% and costs 9.62%.
3. Common stock takes the remaining 48% and costs 12.35%.

Calculate the corporation's weighted-average cost of capital and explain what it means.

TABLE E5.2.1

Source of Capital	% of Capital Structure, w_i	Cost of Capital, k_i	$w_i k_i$
Long-term debt	.38	.0559	.0212
Preferred stock	.14	.0962	.0135
Common stock	.48	.1235	.0593
	100.00		$\sum_1^3 w_i k_i = .094$

$$CC_{wa} = \sum_{i=1}^{3} \square\square\square\square = 9.4\%$$

The weighted-average cost of capital is 9.4%, which means that this corporation would be able to accept all investment projects that would potentially earn returns greater than or at least equal to 9.4%.

2.2.2 RATIO ANALYSIS

Shareholders, creditors, and managers are all very interested in a firm's performance as expressed through its financial statements. Prospective investors as well as current stockholders wish to know more about the firm's trends in returns and potential risks, and ultimately, to have a better understanding of what affects the share price and their share of the firm's profits. Managers' main interest is in their capacity to control and monitor a firm's performance and to take it to the best possible level. All of this would establish the need to analyze the firm's financial statements by the way of constructing and calculating a variety of ratios that would serve as general indicators to assess the firm's performance. Ratio analysis also considers two points of view: the **cross-sectional**, where a comparison of those financial indicators is made at the same point in time, and the **time series**, where the indicators are analyzed as trends extending over a period of time. Most financial ratios are related to the investment, as they are related to the way in which the firm employs and manages its capital. Regarding the term of analysis, most financial ratios are related to short-run analysis, as they address specific aspects of performance, such as the ratios of profitability, liquidity, and operations. In long-run analysis, ratios of debt would be a typical example.

Profitability Ratios

Profitability ratios are also called **efficiency ratios** since the major objective is to assess how efficiently firms utilize their assets and, ultimately, how they are able to attract investers and their capital.

Gross Profit Margin Ratio (GPMR) This ratio shows how much gross profit, GP (sales after paying for the cost of goods sold) is generated by each dollar of net sales, NS (gross sales minus all goods returned).

$$\boxed{\text{GPMR} = \frac{\text{GP}}{\text{NS}}}$$

Example 5.3.1 If gross profit is \$83,420 and net sales is \$185,377, the GPMR would be

$$\text{GPMR} = \frac{\$83{,}420}{\$185{,}377} = .45$$

A gross profit margin of 45% means that out of each dollar of net sales, 45 cents would be gross profit.

Operating Profit Margin Ratis (OPMR) Instead of the gross profit in the GPMR, the OPMR shows the operating profits as they are related to net sales. Operating profit is another term for operating income, which is the same as EBIT (earnings before income and taxes).

$$\boxed{\text{OPMR} = \frac{\text{OY}}{\text{NS}}}$$

Example 5.3.2 If the operating income is \$45,000 the and net sales is \$300,000, the OPMR is

$$\text{OPMR} = \frac{\$45{,}000}{\$300{,}000} = .15$$

which means that 19 cents out of each dollar of net sales in this firm goes to the operating income budget.

Net Profit Margin Ratio (NPMR) This time, the net profit is related to the net sales. The NPMR states how much net profit the firm earns from its volume of sales.

$$\boxed{\text{NPMR} = \frac{\text{NP}}{\text{NS}}}$$

Example 5.3.3 Suppose that the net profit in one firm is \$35,287 and the net sales are \$298,971. The NPMR would be

$$\text{NPMR} = \frac{\$35{,}287}{\$298{,}971} = .118$$

or 11.8%, meaning that of each dollar of net sales this firm would have a little less than 12 cents as net profit. This measure is important especially because it paints a picture of the profit after all expenses, including interest and taxes, have been paid for.

Return on Investment Ratio (ROIR) The ROIR is also known as the ROA (return on assets). It relates net profit (i.e., after interest and taxes) to total assets (TA) of a firm.

$$\boxed{\text{ROIR} = \frac{\text{NP}}{\text{TA}}}$$

Example 5.3.4 Let's use the previous net profit figure of \$35,287 against a total asset of \$250,000. The ROIR would be

$$\text{ROIR} = \frac{\$35{,}287}{\$250{,}000} = 14\%$$

which says that each dollar of the total asset value would give 14 cents in net profit.

Return on Equity Ratio (ROER) In this ratio, the net profit (NP) is related to the owner's equity (OE) in its format of both preferred and common stock. It basically tells stockholders a crucial piece of information: how much of their money a firm would turn into net profit:

$$\boxed{\text{ROER} = \frac{\text{NP}}{\text{OE}}}$$

Example 5.3.5 Suppose that the owner's equity is valued at \$79,500. The net profit of \$35,287 would be forming an ROER as

$$\text{ROER} = \frac{\$32{,}287}{\$79{,}500} = 44\%$$

which tells shareholders that this firm is able to turn 44 cents of each dollar of their investment into a net profit.

Sales – Asset Ratio (SAR) The SAR is another efficiency ratio. It shows how efficient the use of resources is, as an important aspect of a firm's performance and its ability to generate profits. It relates sales to total assets.

$$\boxed{\text{SAR} = \frac{S}{\text{TA}}}$$

Example 5.3.6 Suppose that the volume of sales for a firm reached \$46,890 and its records indicate that the value of its total assets at the beginning of the year was \$60,522 and at the end of the year was \$50,177. The firm's SAR would consist of dividing the sales (S) by the average value of assets since we have two readings:

$$\text{SAR} = \frac{\$46{,}890}{(\$60{,}522 + \$50{,}177)/2} = .85$$

This ratio says that the firm is working hard to put its assets to use in producing and selling its products.

Sales to Net Working Capital Ratio (SNWCR) This time we relate sales to the net working capital, which is basically the firm's short-run net worth or the difference between the current assets and the current liabilities. That current sense of measure is what gives this ratio its more important meanings:

$$\boxed{\text{SNWCR} = \frac{S}{\text{NWC}}}$$

Example 5.3.7 Suppose that the working capital in the firm of Example 5.3.6 is \$3,590. Its SNWCR would be

$$\text{SNWCR} = \frac{\$46{,}890}{\$3{,}590} = 13.1$$

This ratio reflects how the volume of sales relates to the firm's current net worth: in other words, how the net working capital has been put to use.

Market-Based Ratios

Market-based ratios reflect a firm's performance as it is associated with the related market, and therefore the ratios would be looked at with great interest by current investors, potential investors, and managers.

Price – Earnings (P/E) Ratio The P/E ratio is one of the most important and commonly used ratios. It relates the market price of a firm's common stock (MPS) to its earnings per share (EPS).

$$\boxed{\text{P/E} = \frac{\text{MPS}}{\text{EPS}}}$$

This ratio reflects investors' confidence in a firm's financial performance; therefore, the higher the P/E, the higher the appraisal given by the stock market.

Example 5.3.8 If a firm's market price per share of common stock is \$65 and the firm has a \$7.45 earnings per share, the firm's P/E is

$$\text{P/E} = \frac{\$65}{\$7.45} = 8.72$$

which means that this firm's common stock is selling in the stock market for nearly nine times its earnings.

Price – Earnings – Growth Ratio (PEGR) This ratio employs the P/E ratio and relates it to a firm's expected growth rate per year (EGR). It reflects the firm's potential value of a share of stock.

$$\boxed{\text{PEGR} = \frac{\text{P/E}}{\text{EGR}}}$$

Example 5.3.9 Suppose that a firm with a P/E of 8.72 expects an annual growth rate of 8%. Its PEGR would be

$$\text{PEGR} = \frac{8.72}{8} = 1.09$$

It is theorized that PEGRs represent the following:

- If PEGR = 1 to 2: The firm's stock is in the normal range of value.
- If PEGR < 1: The firm's stock is undervalued.
- If PEGR > 2: The firm's stock is overvalued.

Earnings per Share (EPS) The EPS is more important to common stockholders in particular because it is calculated by dividing the net profit (after subtracting the dividends for preferred stock) by the outstanding number of shares of common stock.

$$\boxed{\text{EPS} = \frac{\text{NP} - D_p}{\text{no. shares}}}$$

Example 5.3.10 Suppose that a firm has a net profit of \$600,000. It pays 7% of its dividends to preferred stockholders and distributes the remainder among the 40,000 shares of common stock. Its EPS would be

$$\text{dividends for preferred stocks, } D_p = \$600{,}000 \times .07 = \$42{,}000$$

$$\text{EPS} = \frac{\$600{,}000 - \$42{,}000}{40{,}000} = \$13.95$$

This means that for each share of common stock that investors own, they earn \$13.95.

Dividend Yield (DY) The DY is obtained by dividing dividends of common stock per share (DPS) by the market price of stock (MPS):

$$\boxed{\mathrm{DY} = \frac{\mathrm{DPS}}{\mathrm{MPS}}}$$

If the dividend per share is $1.95 and the stock price is $35, the dividend yield is

$$\mathrm{DY} = \frac{\$1.95}{\$35} = 5.6\%$$

which says that common stockholders receive only 5.6% as a dividend for the price that each share of stock sells for in the market.

Cash Flow per Share (CFPS) The CFPS is just like earnings per share (EPS) except that it uses cash flow instead of net profit. Some financial analysts believe that real operating cash flow (OCF) is a much more reliable measure than net profit, which includes a lot of accounts receivable. A measure of the cash available as related to the number of shares of common stock is a good indicator of a firm's financial health.

$$\boxed{\mathrm{CFPS} = \frac{\mathrm{OCF}}{\text{no. shares}}}$$

Example 5.3.11 Suppose that a firm has an operating cash flow of $65,000 and its shares of common stock reach 500,000 shares outstanding. Its CFPS would be

$$\mathrm{CFPS} = \frac{\$65{,}000}{500{,}000} = \$.13$$

which means that the cash flow per share in this firm is 13 cents.

Payout Ratio (PYOR) This ratio shows how much earnings per share would be paid out as cash dividends for common stockholders *(D_c)*.

$$\boxed{\mathrm{PYOR} = \frac{D_c}{\mathrm{EPS}}}$$

Example 5.3.12 Let's suppose that for a firm with an EPS of $13.95, $3.10 is paid out as a cash dividend per share. The payout ratio would then be

$$\boxed{\mathrm{PYOR} = \frac{\$3.10}{\$13.95} = \$.22}$$

which means that 22 cents out of each dollar earned per share is being paid out as a dividend.

Book Value per Share (BVPS) This ratio shows the stockholder's equity or net worth (NW) for each share held.

$$\boxed{\text{BVPS} = \frac{\text{NW}}{\text{no. shares}}}$$

Example 5.3.13 Suppose that a firm's total assets are $747,000, and total liabilities are $517,000 and there are 20,000 shares outstanding. What would be the book value per share?

$$\begin{aligned} \text{NW} &= \text{A} - \text{L} \\ &= \$747{,}000 - \$517{,}000 \\ &= \$230{,}000 \\ \text{BVPS} &= \frac{230{,}000}{20{,}500} = \$11.50 \end{aligned}$$

This means that each share is worth $11.50 of the firm's net worth.

Price–Book Value Ratio (PBVR) This ratio shows how the market price of a stock (MPS) is related to the book value per share (BVPS):

$$\boxed{\text{PBVR} = \frac{\text{MPS}}{\text{BVPS}}}$$

Example 5.3.14 Suppose that the stock of the firm in Example 5.3.14 is sold in the market for $20. The price–book value ratio would be

$$\text{PBVR} = \frac{\$20}{\$11.50} = \$1.74$$

A PBVR of $1.74 means that this firm is worth 74% more than the shareholders put into it.

Generally, the PBVR can be read like this:

- If PBVR > 1: The firm is utilizing assets efficiently.
- If PBVR < 1: The firm is utilizing assets inefficiently.
- If PBVR = 1: The firm is utilizing on the margin.

Price/Sales (P/S) Ratio This ratio shows how many dollars it takes to buy a dollar's worth of a firm's revenue. It is calculated by dividing the market capitalization (MC), which is (stock price $\times$ no. shares) by the firm's revenue for the last year (TR).

$$\text{MC} = \text{MPS} \times \text{no. shares}$$

$$\boxed{\text{P/S} = \frac{\text{MC}}{\text{TR}}}$$

Example 5.3.15 If we take the stock price and the number of shares from the preceding examples: MPS = \$20 and the number of shares = 20,000, and if we suppose that the revenue of this firm last year was \$650,000, the market capitalization (MC) would be

$$\text{MC} = \$20 \times 20{,}000 = \$400{,}000$$

$$\text{P/S} = \frac{\$400{,}000}{\$650{,}000} = .62$$

A price/sales ratio of 62% is good; it describes a case in which the investors get more than they invest. Generally, market analysts have come up with the criterion that the P/S ratio should be less than if not equal to 75%:

$$\boxed{\text{P/S} \leq .75}$$

and investors should avoid firms with price/sales ratios above 150%.

Tobin's Q-Ratio Tobin's Q-ratio is named after the economist James Tobin, who came up with this ratio as an improvement over the traditional price – book value ratio (PBVR). Tobin believes that both the debt and equity of a firm should be included in the top of the ratio, and instead of depending on the firm's book value the bottom should be the firm's entire assets in their replacement cost, which is adjusted for inflation. In this case, Tobin's Q would reflect accurately where the firm stands.

$$\boxed{\text{Tobin's } Q = \frac{TA_{mv}}{TA_{rv}}}$$

where TA_{mv} is the market value of the firm's total assets and TA_{rv} is the replacement value of total assets.

Example 5.3.16 If the market value of total assets of a firm is \$127 million and its replacement cost is \$150 million, its Tobin's Q value would be

$$\text{Tobin's } Q = \frac{127}{150} = 84.6\%$$

Tobin referred to the rule of thumb for this ratio:

- If Tobin's $Q > 1$: The firm would have the capacity and incentive to invest more.
- If Tobin's $Q < 1$: The firm cannot invest and may acquire assets through merger.

Operational Ratios

Members of this group of ratios are also called **activity ratios**. They deal with the extent to which the firm is able to convert various accounts into cash or sales. These accounts include inventory, accounts receivable, accounts payable, fixed assets, and total asset turnover.

Inventory Turnover Ratio (ITR) This ratio relates the cost of goods sold (COGS) to the value of inventory (INY):

$$\boxed{\text{ITR} = \frac{\text{COGS}}{\text{INY}}}$$

Often, inventory is calculated as an average of the inventory at the beginning of the year and at the end of the year.

Example 5.3.17 If the cost of goods sold is \$130,000 and the average value of inventory is \$53,560, the ITR would be

$$\text{ITR} = \frac{\$130{,}000}{\$53{,}560} = 2.43$$

An ITR of 2.43 means that the firm moves its inventory 2.43 times a year. The ITR can also be expressed as the **average age of inventory** (AAINY), which is a measure of how many days the average inventory stays in stock. That would be done by dividing the number of days a year (365) by the ITR.

$$\boxed{\text{AAINY} = \frac{365}{\text{ITR}}}$$

So if we divide 365, by 2.43 we get

$$\text{AAINY} = \frac{365}{2.43} = 150$$

which means that it would take 150 days for this firm to carry its inventory.

Accounts Receivable Turnover (ART) The ART is also called the **average collection period**, which shows the extent to which customers pay their credit bills. It is the account receivable (AR) divided by the average daily sales (DS):

$$\boxed{\text{ART} = \frac{\text{AR}}{\text{DS}}}$$

Example 5.3.18 If the account receivable has \$550,000 and the annual sales are \$3,650,000, we can get ART by first getting the daily sales by dividing the annual sales by 365:

$$\frac{\$3{,}650{,}000}{365} = 10{,}000$$

$$\text{ART} = \frac{\$550{,}000}{10{,}000} = 55$$

which means that it would take the firm 55 days to collect its bills. This is not good unless the firm has a 60-day collection standard, but it is usually 30 days.

Account Payable Turnover (APT) The APT is also called the **average payment period**. It is similar to the accounts receivable turnover in that it divides the account payable (APY) by the average daily purchase (DP).

$$\boxed{\text{APT} = \frac{\text{APY}}{\text{DP}}}$$

Example 5.3.19 Suppose that a firm's accounts payable shows \$480,000 and its daily purchases are estimated by \$15,517. Its APT would be

$$\text{APT} = \frac{\$480{,}000}{\$15{,}517} = 31 \text{ days}$$

That means that on average the firm has 31 days to pay its bills, which would be a very good standard.

Fixed Asset Turnover (FAT) The FAT relates the volume of sales (NS) to the firm's fixed assets (FA).

$$\boxed{\text{FAT} = \frac{\text{NS}}{\text{FA}}}$$

Example 5.3.20 Suppose that a firm has a total value of fixed assets of $79,365 and its net sales are estimated at $133,773. Its FAT value would be

$$\text{FAT} = \frac{\$133{,}773}{\$79{,}365} = 1.7$$

which means that this firm is able to generate a sales value 1.7 times that of the value of its fixed assets.

Total Asset Turnover (TAT) The TAT is just like the FAT except that this time the net sales value is related to all assets in the firm instead of only the fixed assets.

$$\boxed{\text{TAT} = \frac{\text{NS}}{\text{TA}}}$$

Example 5.3.21 Suppose that all assets in Example 5.3.20 are $140,593; the total asset turnover TAT would then be

$$\text{TAT} = \frac{\$133{,}773}{\$140{,}593} = .95$$

A total asset turnover of 95% means that a firm is able to turn over 95% of its asset value in net sales.

Liquidity Ratios

Liquidity ratios show a firm's ability to handle and pay its short-term liabilities and obligations. The more liquid assets the firm can lay its hands on, the easier and smoother the entire performance will be. Liquidity ratios include the current ratio, the quick ratio, the net working capital ratio, and the cash ratio.

Current Ratio (CR) This ratio is probably the most popular among financial ratios for its direct relevance. It simply describes how current assets (CA) are related to current liabilities (CL):

$$\boxed{\text{CR} = \frac{\text{CA}}{\text{CL}}}$$

Example 5.3.22 Suppose that a firm's current assets are valued at $1.5 million and its current liabilities are estimated at $980,711. The firm's current ratio would be

$$\text{CR} = \frac{\$1{,}500{,}000}{\$980{,}711} = \$1.53$$

A current ratio of 1.53 means that this firm has 1 dollar and 53 cents in its current asset value to meet each dollar of its current obligations. Generally, the current ratio is recommended by most financial analysts to be 2 or more, which means that for a firm to be robust, it has to own at least twice as much as it owes.

$$\boxed{\mathrm{CR} \geq 2}$$

The firm here has to dedicate 65 cents out of each dollar of its current assets to pay its current creditor's claims $(1/1.53) = .65$.

Acid-Test Ratio (QR) This ratio is also called the **quick ratio**. It is similar to the current ratio above except that the value of inventory is taken away from the current assets.

$$\boxed{\mathrm{QR} = \frac{\mathrm{CA} - \mathrm{INY}}{\mathrm{CL}}}$$

Example 5.3.23 If the entire inventory in the firm of Example 5.3.22 was estimated at \$380,664, the quick ratio would be

$$\mathrm{QR} = \frac{\$1{,}500{,}000 - \$380{,}664}{\$980{,}711} = 1.14$$

which means that the firm has a dollar and 14 cents for each dollar of its creditor's claims. It is noteworthy to mention here that if there are any prepaid items, they would also be subtracted from the current assets along the inventory value.

Net Working Capital Ratio (NWCR) The net working capital is the short-run net worth of a firm. It is the difference between the current assets and the current liabilities. If we divide the net working capital (NWC) by the available total assets (TA), we get the net working capital ratio (NWCR), which shows the firm's potential cash capacity.

$$\boxed{\mathrm{NWCR} = \frac{\mathrm{NWC}}{\mathrm{TA}}}$$

Example 5.3.24 Suppose that the total assets for the firm in Example 5.3.23 is \$49,950,592 and that the current assets and liabilities stay at \$1,500,000 and \$980,771, respectively. The firm's net working capital would be

$$\begin{aligned} \mathrm{NWC} &= \mathrm{CA} - \mathrm{CL} \\ &= \$1{,}500{,}000 - \$980{,}711 \\ &= \$519.289 \end{aligned}$$

and its net working capital ratio would be

$$\text{NWCR} = \frac{\$519{,}289}{\$4{,}950{,}592} = .10$$

which means that this firm has 10 cents in current net worth in each dollar of its total assets.

Cash – Current Liabilities Ratio (CCLR) This ratio tracks down cash and marketable securities *(*C + MS*)* that are at hand and weighs them against the due current obligations and liabilities (CL).

$$\boxed{\text{CCLR} = \frac{\text{C} + \text{MS}}{\text{CL}}}$$

Example 5.3.25 If we keep the current liabilities of the last firm at \$980,711 and assume that cash is counted as \$27,500 and marketable securities estimated at \$31,342, the firm's cash-to-current liabilities ratio (CCLR) would be

$$\text{CCLR} = \frac{\$27{,}500 + \$31{,}342}{\$980{,}711} = .06$$

which says that this firm holds some liquid assets in terms of cash and marketable securities equal to 6 cents to meet each dollar of its current liabilities.

Interval Ratio (InR) This ratio is another expression of the cash – current liabilities ratio but in terms of time. It reveals how many days a firm is able to meet its short-term obligations. It is obtained by dividing not only cash and marketable securities, but also accounts receivable (AR), by the daily expenditures on current liabilities or current liabilities per day (CLPD).

$$\boxed{\text{InR} = \frac{\text{C} + \text{MS} + \text{AR}}{\text{CLPD}}}$$

Example 5.3.26 Let's consider \$18,500 in accounts receivable in Example 5.3.25. Also consider that the average daily expenditures on obligations is calculated at \$1,250.

$$\text{InR} = \frac{\$27{,}500 + \$31{,}342 + \$18{,}500}{\$1{,}250} = 62 \text{ days}$$

An interval ratio of 62 days means that the firm can continue to meet its average daily spending of \$1,250 on obligations for 2 months, tapping its reserve of cash, marketable securities, and accounts receivable.

Debt Ratios

Debt ratios are also called **leverage ratios**. Because of the increased financial leverage and risk that comes with using more debt in a firm's financing, debt ratios have more importance. These ratios indicate the extent to which a firm's assets are tied to a creditor's claims and therefore the firm's ability to meet the fixed payments that are due to pay off debt.

Debt – Asset (D/A) Ratio This is a direct measure of the percentage of a firm's total assets that belong to creditors: in other words, how much of other people's money is used to generate business profits. It is obtained simply by dividing total liabilities or debt (TD) by total assets (TA).

$$\boxed{\mathrm{D/A} = \frac{\mathrm{TD}}{\mathrm{TA}}}$$

Example 5.3.27 If a firm has a total debt of \$734,000 and its total assets are estimated at \$1,930,570, its debt – asset ratio would be

$$\mathrm{D/A} = \frac{\$734{,}000}{\$1{,}930{,}570} = .38$$

which means that 38% of the firm's assets is financed with debt.

Debt – Equity (D/E) Ratio This ratio weighs a firm's total debt (TD) to its owner's equity (E). It shows the percentage of owner's equity that is generated by debt.

$$\boxed{\mathrm{D/E} = \frac{\mathrm{TD}}{E}}$$

Example 5.3.28 If the firm in Example 5.3.27 has an equity estimated at \$1,200,000, its D/E ratio would be

$$\mathrm{D/E} = \frac{\$734{,}000}{\$1{,}200{,}000} = .61$$

A D/E of 61% means that for every dollar of owner's equity in the firm, 61 cents is owed to creditors.

Solvency Ratio (Sol) The solvency ratio is the opposite of the D/A ratio: It is obtained by dividing total assets by total debt. It shows to what extent a firm's total assets can handle its total liabilities or debt.

$$\boxed{\text{Sol} = \frac{\text{TA}}{\text{TD}}}$$

Example 5.3.29 Let's reverse the D/A ratio in Example 5.3.27, and see what kind of solvency ratio we get:

$$\text{Sol} = \frac{\$1{,}930{,}570}{\$734{,}000} = 2.6$$

This solvency ratio means that the firm actually owns 2.6 times more than it owes, and therefore it is solvent. Solvency criteria are:

- If Sol > 1: The firm is solvent.
- If Sol < 1: The firm is insolvent.
- If Sol = 1: The firm is on the margin when its total debt is equal its total assets.

Times Interest Earned Ratio (TIER) This ratio measures the extent to which a firm is able to make its interest payments. It is obtained by dividing the firm's operating income (OY) (earnings before interest and taxes) by the annual amount of interest due to creditors.

$$\text{TIER} = \frac{OY}{I}$$

Example 5.3.30 Suppose that a firm's operating income is $170,000 and its total annual interest payment is $35,000. The times–interest earned ratio would be

$$\text{TIFR} = \frac{\$170{,}000}{\$35{,}000} = 4.9$$

This means that this firm has an operating income almost five times larger than the interest payment due. We can also say that for every dollar of interest the firm pays to creditors, it has almost $5 in the form of operating income. Still yet, we can say that the firm has enough of a paying capacity to be able to service its debt for about five years.

Operating Income – Fixed Payments Ratio (OYFPR) This ratio is an expanded TIER. Instead of interest payments only in the denominator, all other fixed payments are added to the interest payments, such as payment for principal (P), payments for preferred stocks as dividends (D_{ps}), and scheduled lease payments (L).

$$\boxed{\text{OYFPR} = \frac{\text{OY}}{I + P + D_{ps} + L}}$$

Example 5.3.31 Consider the following fixed payments as additions to the interest payment in Example 5.3.30: P, \$22,000; D_{ps}, \$51,000; L, \$11,000. Then the (OYFPR) would be

$$\begin{aligned}\text{OYFPR} &= \frac{\$170{,}000}{\$35{,}000 + \$22{,}000 + \$51{,}000 + \$11{,}000}\\ &= 1.43\end{aligned}$$

Still, this firm's operating income is 1.43 times more than all the fixed payments due.

Note that the principal payment, lease payment, and preferred stock payment have to be in before-tax status. If they are in after-tax status, they have to be converted to before-tax status by dividing them by $(I - T)$.

$$(P + D_{ps} + L)_b = \frac{(P + D_{ps} + L)_a}{(I - T)}$$

where T is the corporate tax rate.

2.2.3 THE DuPONT MODEL

The **DuPont model** is a system of financial analysis that has been used by financial managers since its invention by financial analysts working at the DuPont Corporation in the 1920s. It can be described as a collective method of financial analysis, although it has been characterized by some analysts as a complete system of financial ratio utilization. The basic premise of this model is to combine the firm's two financial statements:

1. The income–expense statement
2. The balance sheet

and to incorporate the impact of three important elements:

- The profits on sales, represented by the net profit margin ratio
- The efficiency of asset utilization, represented by the total asset turnover ratio
- The leverage impact, represented by the equity multiplier

The model has two major objectives:

1. To analyze what determines the size of return that investors look forward to receiving from firms in which they invest. This objective is achieved by breaking the return on equity (ROE) into two components: the return on investment (ROI) and the equity multiplier (EM).

$$\boxed{\mathrm{ROE} = \mathrm{ROI} \cdot \mathrm{EM}}$$

2. To break the elements of ROE into subelements: The return on investment is obtained by multiplying the net profit margin (NPM) by the total assets turnover (TAT):

$$\boxed{\mathrm{ROI} = \mathrm{NPM} \cdot \mathrm{TAT}}$$

Furthermore, the net profit margin is obtained by dividing net profits by net sales, and total asset turnover is obtained by dividing net sales by total assets:

$$\mathrm{NPM} = \frac{\mathrm{NP}}{\mathrm{NS}} \quad \text{and} \quad \mathrm{TAT} = \frac{\mathrm{NS}}{\mathrm{TA}}$$

The equity multiplier (EM) is the ratio of total assets to owner's equity:

$$\mathrm{EM} = \frac{\mathrm{TA}}{\mathrm{OE}}$$

To substitute all of the elements, we get

$$\mathrm{ROE} = \frac{\mathrm{NP}}{\mathrm{NS}} \cdot \frac{\mathrm{NS}}{\mathrm{TA}} \cdot \frac{\mathrm{TA}}{\mathrm{OE}}$$

Canceling out NS and TA, we get:

$$\boxed{\mathrm{ROE} = \frac{\mathrm{NP}}{\mathrm{OE}}}$$

Figure 5.1 shows how the various elements are taken from two financial statements, the balance sheet and the income – expense statement, to establish the return on equity.

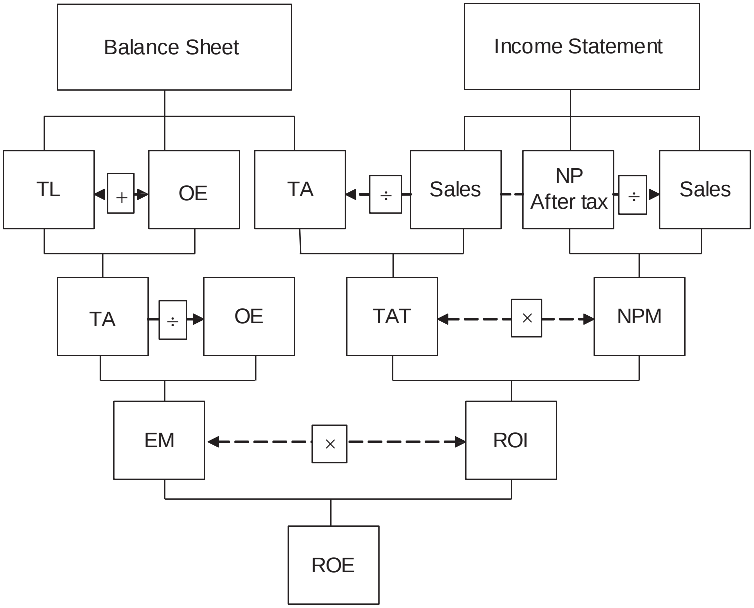

FIGURE 5.1 The DuPont Model.

2.2.4 A FINAL WORD ABOUT RATIOS

We have discussed a large number of ratios over five categories covering almost every possible aspect of business performance. These ratios are not to be memorized but to be understood and used and interpreted well. They are mathematical terms of one amount divided by another and therefore must be understood as such. The interpretation simply has to be focused on reading the numerator as part of the denominator or the denominator as the whole, including the numerator: simply how the part above the division line relates to the part below the line. Business performance has many aspects, and this is a reason to say that an analyst would be much wiser to use many ratios than only one or two. A comparison has to be consistent in terms of the time period, and requires consistency in size and line of product, among many other factors. A comparison can be made horizontally by the cross-section approach to compare the same ratio across firms, and vertically by the time-series approach to compare ratios of the same firm over the years. Data have to be from sources that were already checked and approved and should be from audited statements. Because of many overlaps, ratios for particular purposes of financial analysis have to be chosen carefully to prevent redundancy.

Summary

In this unit four major financial securities were discussed in detail as to matters related to their calculations. Stocks, bonds, mutual funds, and options are the fundamental tools of investment, in addition to an integrating topic as to the cost of capital and ratio analysis.

Types of stocks were discussed briefly, and the process of buying and selling stocks was illustrated with examples. The focus was on common stocks as to their evaluation, the cost of new issues, their value with two-stage dividend growth, their cost through the CAMP model, and other methods of evaluation, such as the P/E multiples method and the liquidation value per share method. Valuation and cost of preferred stocks followed.

The other important security was bonds. The discussion followed a similar pattern, starting with bond evaluation, premium, and discount prices, followed by illustrations of premium amortization and discount accumulation. Several examples explained what happened when bonds were purchased between interest days and how the yield rate was established by the average method as well as by the interpolation and current yield methods. This issue of duration concluded the discussion of bonds.

The third major security is the mutual fund, a combination of stocks and bonds but with their own character and therefore worthy of separate consideration. Discussed were fund evaluation and loads, which are commissions on the purchase and sale of funds. Also discussed was the performance measures of four major criteria: the expense ratio, the total investment expense, the reward – variability ratio, and Treynor's index. The bonds chapter concluded with a related subject on systematic risk, where beta and alpha were explained.

The last major security discussed in this unit was options. We started with a discussion of the choices available to an option holder: whether he would or would not exercise his rights to buy or sell, or even choose to skip action and let the time run out and the rights expire. This discussion was followed by a detailed explanation of the dynamics of making profits with options through buying and selling calls and puts. As for the value of options, the intrinsic value of calls and puts was explained with examples and a graphical presentation. Also, the time value of calls and puts and the determinants of option value were addressed. Finally, the option valuations were put together and the combined intrinsic values were calculated.

The last chapter in this unit covered cost of capital and ratio analysis. The cost of capital is a central issue in investment and in finance in general, and that was why it deserved its room in this unit, especially what the cost of capital meant, first, as a rate of return before and after taxes, and then as a weighted average.

In the ratio analysis section we explained many financial ratios with solved examples to emphasize their direct meanings and how they can serve as indicators for financial performance. This topic specifically and the unit generally concluded with the DuPont model, which wrapped up most of the concepts in financial analysis.

List of Formulas

Stocks

Market capitalization rate (expected rate):

$$\mathrm{MCR} = \mathrm{Er} = r = \frac{D + P_1 - P_0}{P_0}$$

Current price:

$$P_0 = \frac{D_1 + P_1}{1 + r}$$

Next year's price:

$$P_1 = \frac{D_2 + P_2}{1 + r}$$

Collective current price:

$$P_0 = D_1 \sum_{t=1}^{\infty} \frac{1}{(1 + r)^t}$$

Current price when dividends grow at a constant rate:

$$P_0 = \frac{D_0(1 + g)}{r - g}$$

Market capitalization rate when dividends grow at a constant rate:

$$r = \frac{D_1}{P_0} + g$$

Two-state dividend growth:

$$P_0 = D_0 \left[\frac{1 - ((1 + g_1)/(1 + r)^n)}{r - g_1} \right] (1 + g_1) + D_{n+1} \left(\frac{1}{r - g_2} \right) \left(\frac{1}{1 + r} \right)^n$$

Cost of common stock—CAMP model:

$$r_c = R_f + \beta(R_m - R_f)$$

Value of common stock by P/E multiples:

$$V_c = (\mathrm{EPS}_e)(\mathrm{P/E}_i)$$

Value of preferred stock:

$$P_p = \frac{D_p}{r_p}$$

Cost of preferred stock:

$$r_p = \frac{D_p}{N_p}$$

Bonds

Value of bond:

$$B_0 = I \left[\sum_{t=1}^{n} \frac{1}{(1+i)^t} \right] + M \left[\frac{1}{(1+i)^n} \right]$$

$$B_0 = I(\mathrm{PV/FA}_{i,n}) + M(\mathrm{PVIF}_{i,n})$$

$$B_0 = I(a_{\overline{n}|i}) + M(v^n)$$

Discount price:

$$D_s = M(i - r)a_{\overline{n}|i}$$

Premium price:

$$P_m = M(r - i)a_{\overline{n}|i}$$

Price between interest dates:

$$B_{\mathrm{bd}} = B_0 + Y_{\mathrm{bp}}$$

$$B_{\mathrm{bd}} = B_0[1 + i(\mathrm{bp})]$$

Price quoted:

$$B_q = B_{\mathrm{bd}} - I_{\mathrm{ac}}$$

Yield rate:

$$\mathrm{YR} = \frac{\mathrm{AII}}{\mathrm{AAI}}$$

$$\mathrm{AII} = Mr - \frac{P_m}{n}$$

$$\text{AII} = Mr + \frac{D_s}{n}$$

$$\text{AAI} = \frac{M + B}{2}$$

Current yield:

$$\text{YR} = \text{Cr}(2 + \text{Cr})$$

$$\text{Cr} = \frac{I - P_m/n}{B_q}$$

Duration:

$$D = \frac{\sum_{t=1}^{n} \frac{t}{(1+i)^t} + \frac{nM}{(1+r)^n}}{\sum_{t=1}^{n} \frac{t}{(1+i)^t} + \frac{nM}{(1+i)^n}}$$

Percent change in bond price:

$$\%\Delta B = \Delta i(\text{VL})$$

$$\%\Delta B = \Delta i\left(\frac{D}{1+i}\right)$$

Mutual funds

Net asset value:

$$\text{NAV} = \frac{1}{S}[(\text{MV} + C) - O]$$

Purchase price:

$$\text{PP} = \text{NAV}\left(\frac{1}{1 - L_F}\right)$$

Selling price:

$$\text{SP} = \text{NAV}(1 - L_B)$$

Expense ratio:

$$\text{ER} = \exp\left[\frac{1}{(A_1 + A_2)/2}\right]$$

Total investment expense:

$$\text{TIE} = \frac{1}{n}(L_F + L_B) + \text{ER}$$

Reward variability ratio:

$$\text{RVR} = \frac{1}{\sigma_j}(R_{jt} - R_{ft})$$

Rate of return:

$$R = \frac{S}{1\text{F}}(.6.P + D + G)$$

Treynor index:

$$\text{TI} = \frac{1}{\beta_j}(R_{jt} - R_{ft})$$

Specific fund beta:

$$\beta_i = \frac{\text{Cov}(R_i, R_m)}{\sigma_m^2}$$

Market beta:

$$\beta_m = \frac{\sigma_m^2}{\sigma_m^2} = 1$$

Collective beta:

$$\beta_p = \sum_{j=1}^{k} \beta_j w_j \qquad j = 1, 2, \ldots, k$$

Options

Intrinsic value of a buyer's call:

$$\text{IVC}_B = \max[(\text{MP} - \text{SP}), 0]$$

Intrinsic value of a writer's call:

$$\text{IVC}_W = \min[(\text{SP} - \text{MP}), 0]$$

Intrinsic value of a buyer's put:

$$\text{IVP}_B = \max[(\text{SP} - \text{MP}), 0]$$

Intrinsic value of a writer's put:

$$\text{IVP}_w = \min[(\text{MP} - \text{SP}), 0]$$

Time value of a call:

$$\mathrm{TV}_c = \mathrm{OP} - \mathrm{IVC}$$

Time value of a put:

$$\mathrm{TV}_p = \mathrm{OP} - \mathrm{IVP}$$

Delta ratio:

$$D = \frac{\Delta \mathrm{OP}}{\Delta \mathrm{MP}}$$

Call value:

$$\mathrm{VC} = \mathrm{MP}[N(d_1)] - (e)^{-rt} \cdot \mathrm{SP}[\mathrm{N}(d_2)]$$

Put value:

$$\mathrm{VP} = [\mathrm{VC} + \mathrm{SP}(e)^{-rt}] - \mathrm{MP}$$

$$d_1 = \frac{\ln(\mathrm{MP}/\mathrm{SP}) + T(r + \sigma^2/2)}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Cost of capital

Cost of capital for bonds:

$$\mathrm{CC}_b = \frac{I + [(\mathrm{M} - \mathrm{NP})/n]}{(\mathrm{NP} + \mathrm{M})/2}$$

After-tax cost of capital:

$$\mathrm{CC}_a = \mathrm{CC}_b(1 - T)$$

Weighted-average cost of capital:

$$\mathrm{CC}_{wa} = \sum_{i=1}^{n} w_i k_i$$

Exercises

1. A man invested in 500 shares of a local business stock selling for \$15.75 a share. A while later he sold half of his shares at \$17.25. Calculate his investment rate of return and his capital gain.

2. An investor pays \$65 plus .00675 on her investment of \$3,000. She plans to purchase whatever she can get of the stock of Exercise 1 priced at \$15.75. Find the net investment and how many shares she is able to purchase. Also calculate the yield on investment.

3. Find the expected rate of return for an investor who purchases 130 shares of stocks at \$33.50 per share with dividends expected to be \$1.95 per share. Assume that he would sell 80 shares at \$35.00.

4. A company is selling its stock at \$17.95 per share and expecting it to grow by 5%. It usually distributes 50% of its earnings per share as dividends. Find the value of this stock if the earning per share is \$2.15 and if an investor would like to earn a 9% yield.

5. A company decides to sell a new issue of its original stock of \$25 per share by lowering the price by 10%. What would be the cost of the new stock if the dividend is \$2.15 and the flotation cost is \$.50? Assume that the stock grows by 4.5%.

6. An investor is contemplating investing heavily in a stock whose dividend is \$27 per share but expected to grow by 18% for the next 3 years and by 12% after that. Find the current price of this stock if this investor requires a rate of return of at least 15%.

7. If the price/earnings ratio of a textile industry is $7\frac{1}{2}$, estimate the value of a share in an apparel firm whose earning per share is \$4.15.

8. If the required rate of return is $9\frac{3}{3}\%$ and the dividend for preferred stock is \$6.50, what would be the value of this preferred stock?

9. Find the cost of preferred stock in Exercise 8 if the value of the stock dips to \$50 and the flotation cost is \$1.75.

10. What is the purchase price of a \$1,000 bond that is maturing in 20 years at 12% interest if the required rate of return is 15%?

11. An investor who wants to yield of 8% would like to invest in a $1,000 bond maturing in 15 years at a coupon rate of 10.5%. Find the bond's current value.

12. A $2,000 bond is redeemable at a coupon rate of 6.5% in 10 years. Would you purchase it at premium or discount price if you want it to yield 8%?

13. What if another investor requires only a 5% yield. Would he buy the bond of Exercise 12 at a premium or a discount price?

14. A bond has a face value of $2,000 redeemable in 5 years at a coupon rate of 8%. Construct the premium amortization schedule if the bond is to be purchased to yield 6%.

15. Consider a $3,000 bond with a coupon rate of 7% but purchased to yield $8\frac{1}{2}$%. Construct the discount accumulation schedule for its maturity period of 5 years.

16. A $5,000 bond with a semiannual coupon at $8\frac{1}{2}$% is redeemable at par value on November 1, 2015. Find the purchase price on July 15, 2013 to yield $7\frac{1}{2}$%.

17. A $4,000 bond redeemable at $6\frac{1}{2}$% in $7\frac{1}{2}$ years. Find (**a**) the flat price to yield 8%, (**b**) the bond net price, and (**c**) the seller's share of the accrued interest.

18. Find the yield rate for a bond purchased at a quoted price of $2,250 redeemable in 10 years. The face value is $2,000 at a coupon rate of $6\frac{1}{4}$%.

19. Calculate the duration of a bond with a par value of $2,000 redeemable in 7 years at a coupon rate of $8\frac{1}{4}$% when the market yield is 11%.

20. Find the volatility factor (VL) in Exercise 19 and explain what it means for the bond price.

21. Calculate the net asset value of the mutual fund of fire securities with the following information:

Security	No. Shares	Share Price ($)	Liabilities ($)	Cash ($)	Outstanding Shares
A	337,000	10	538,000	444,500	500,000
B	250,000	15			
C	118,500	25.50			
D	120,095	31.95			
E	85,000	23			

22. If the net asset value of a mutual fund is $31.75, the front-end load is 5%, and the back-end load is $5\frac{1}{2}$%, find the purchase and sale prices.

23. Find the total investment expense for the fund in Exercise 22 if it is held for 5 years given the expense ratio of .075%.

24. If the standard deviation of a fund returns is 13.35 and the return is 6% but the risk-free return in the market is 4.5%, find the reward-to-variability ratio.

25. Suppose that the market beta is 2.05. Find Treynor's index for the fund in Exercise 22 and explain what it means.

26. Find the intrinsic value of a call for both the buyer and writer if the market price of the underlying stock is $37.50 per share, up from $34.00 three months ago.

27. Find the buyer's and writer's intrinsic value of a put whose strike price is $28.50 when the market price of the stock goes up to $32.00.

28. Suppose that the market price of a stock is up to $37.00 and an investor has $550 worth of 100 shares with the strike price of a call at $33.00. Calculate the time value for a buyer's call.

29. If the time value of an option is $4.60 and the cost of 100 shares is $360, what would be the intrinsic value of the option?

30. Suppose that the market price of an underlying stock went from $40.00 to $45.00 and the option price went from $8.00 to $10.00. Find the delta ratio. Explain what it means.

31. If the strike price of a call option is $75.00 with a 6-month maturity and the market price of the underlying stock is $88.00 with a standard deviation of returns of 2.34, what is the value of the call if the risk-free rate is 6%?

32. For a financial operation involving selling bonds at a 10% discount with a flotation cost of 3%, what would be the before- and after-tax costs of capital if the par value of the coupon is $2,000 at a coupon rate of 7% and a redemption period of 15 years?

33. Calculate the weighted-average cost of capital for a firm with the following capital, weight, and cost of capital for the individual sources.

Capital Source	Percent of Capital	Cost of Individual Source ($)
Preferred stocks	17	12.77
Common stocks	60	15.56
Long-term debt	23	10.57

UNIT III

Mathematics of Return and Risk

3.1 Measuring Return and Risk

Financial securities such as stocks and bonds, as well as other investment assets, have to be evaluated to determine how good an investment they may offer. Such a process of valuation presents the opportunity to link and assess two of the most significant determinants of the security share price: risk and return, whose assessment is the core of all major financial decisions. **Risk**, in its most fundamental meaning, refers to the chance that an undesirable event will occur. In a financial sense, it is defined as the chance to incur a financial loss. When an asset or an investment opportunity is dubbed as "risky," it would be thought to have a stronger chance of bringing a financial loss. It would refer implicitly to the variability of the returns of that asset. Conversely, the certainty of the return, such as the guaranteed return on a government bond, would refer to a case of no risk. We can say further that the investment risk refers to the probability of having low or negative returns on invested assets, such that the higher the probability of getting a low or negative return on an asset, the riskier that investment would be.

The **return** on an investment asset is defined by the change in value, in addition to any cash distribution, all expressed as a percentage of the asset original value. For example, if 100 shares of stock are purchased at $15 per share and sold for $17 per share, the change in value would be $200 ($1,700 − $1,500), and if within the period between purchase and sale, $75 was received in dividends, then both the change in value, $200, and the cash dividend, $75, would be divided by the original value of the investment ($1,500), to get the return, 18.3%:

$$\text{return} = \frac{\$275}{\$1{,}500} = 18.3\%$$

3.1.1 EXPECTED RATE OF RETURN

The **expected rate of return** (k_e) is the sum of products of individual returns (k_i) and their probabilities (Pr_i). It is, therefore, a weighted average of returns.

$$k_e = \sum_{i=1}^{n} ki.\,\mathrm{Pr_i}$$

$$k_e = k_1 \cdot \mathrm{Pr}_1 + k_2 \cdot \mathrm{Pr}_2 + \cdot\ \cdot\ \cdot + k_n \cdot \mathrm{Pr}_n$$

TABLE 1.1

Asset X	k_i	$\Pr_i$	$k_i \cdot \Pr_i$
k_1	.09	.45	.0405
k_2	.10	.30	.03
k_3	.11	.25	.0275
			$\sum k_i \Pr_i = .098$

Let's calculate the expected return on asset X if there are three probable returns: 9% at 45% probability, 10% at 30% probability, and 11 % at 25% probability (see Table 1.1):

$$k_e = k_1 \cdot \Pr_1 + k_2 \cdot \Pr_2 + k_3 \cdot \Pr_3$$
$$= .0405 + .03 + .0275$$
$$= .098 \text{ or } 9.8\%$$

3.1.2 MEASURING THE RISK

The first simple and straightforward way to measure the risk of an asset is the range of returns or the **dispersion**, which is the difference between the highest and the lowest returns. If we take asset X above, which has three probable returns, 9%, 10%, and 11%, if we compare it to asset Y, which also has three probable returns, 5%, 10%, and 15%, and if we calculate the ranges of both sets of returns, the X range $= 2(11-9)$ and the Y range $= 10(15-5)$ (see Table 1.2), we can say that asset Y is riskier than asset X because the range of Y returns is larger. The range reflects the variability, which stands for the risk. We can conclude that the greater the range of returns of an asset, the more the variability and the higher the risk.

Building on the variability notion, the second measure of risk can be the probability distribution of returns. The tighter the probability distribution, the more likely that the actual return will be close to the value expected and therefore have a lower risk; and vice versa. The wider the probability distribution, the higher the variability and therefore the higher the risk.

TABLE 1.2

k_i	X (%)	Y (%)
k_1	9	5
k_2	10	10
k_3	11	15
Range	$11 - 9 = 2$	$15 - 5 = 10$

TABLE 1.3

	k_i	k_e	$k_i - k_e$	$(k_i - k_e)^2$	Pr_i	$(k_i - k_e)^2 \cdot Pr_i$
Asset X						
k_1	.09	.098	–.008	.000064	.25	.000016
k_2	.10	.098	.002	.000004	.50	.000002
k_3	.11	.098	.012	.000144	.25	.000036
				$\sum_{i=1}^{3}(k_1 - k_e)^2 \cdot Pr_i$		.000054
Asset Y						
k_1	.05	.01	−.05	.0025	.25	.000625
k_2	.10	.01	0	0	.50	0
k_3	.15	.10	.05	.0025	.25	.000625
				$\sum_{i=1}^{3}(k_1 - k_e)^2 \cdot Pr_i$		.00125

Standard Deviation (σ)

The standard deviation (σ) would be an appropriate tool to measure the dispersion around the expected return; that is, the higher the standard deviation, the wider the dispersion and the greater the risk. Let's assign some probabilities, such as 25%, 50%, and 25%, to the last sets of returns for assets X and Y (see Table 1.3 and Figure 1.1). and let's calculate the standard deviations.

$$\text{standard deviation } \sigma = \sqrt{\textstyle\sum_{i=1}^{3}(ki - ke)2\, Pri}$$

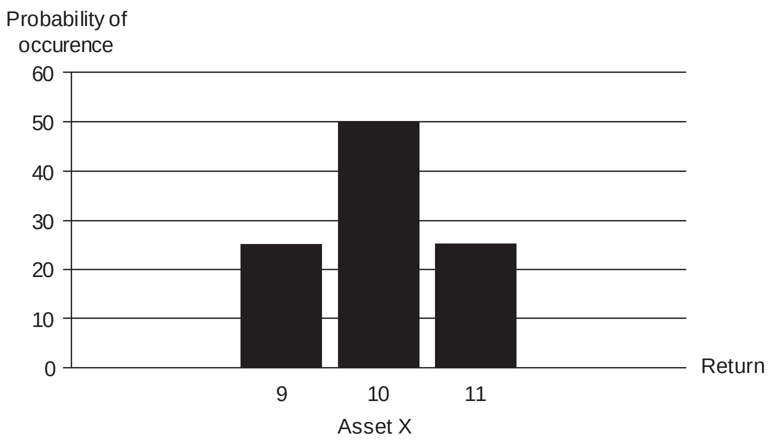

FIGURE 1.1a

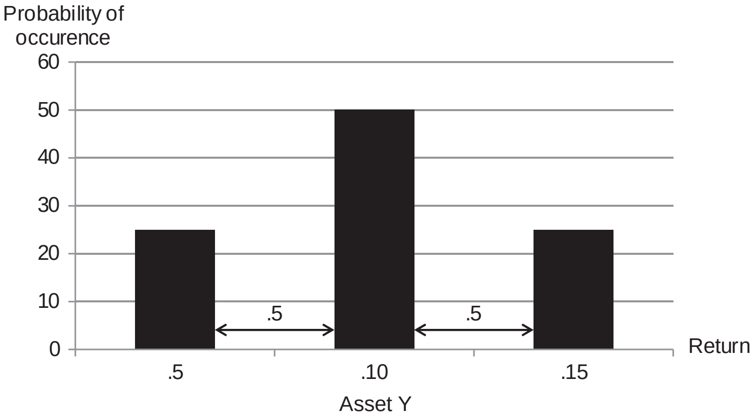

FIGURE 1.1b

$$\sigma_x = \sqrt{.000054} = .0073$$
$$\sigma_y = \sqrt{.00125} = .0353$$

So the standard deviation for the returns on asset Y (σ_y) is larger than the standard deviation of the returns on asset X, which makes asset Y riskier than asset X. In other words, the returns on asset X are closer to their own expected value than are the returns on asset Y to their expected value. Figures 1.1 and 1.2 show this fact visually.

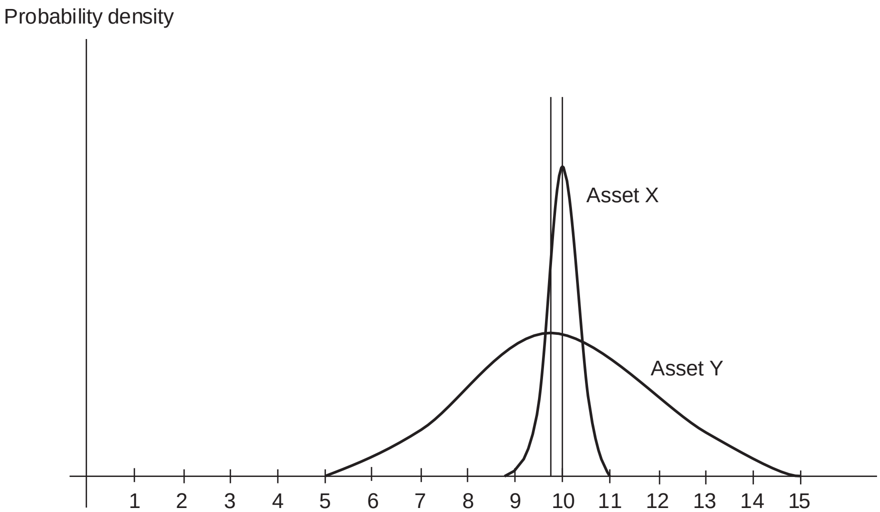

FIGURE 1.2

Furthermore, if we assume that the probability distribution is normal, this would mean that the expected return on asset X (9.8%) would, in fact, be within $\pm$ 1 standard deviation 68.26% of the time: that is, between 9.07% (.098 $-$ .0073), and 10.53 (.098 $-$ .0073). It would also be within $\pm$ 3 standard deviations .0219 (3 $\times$ 0073) 99.74% of the time: that is, between 7.61 (.098 $-$ .0219) and 11.99 (.098 $+$ .0219).

There is another more reliable measure of risk than the standard deviation, especially when the expected returns of the assets are not equal, as in the example above where they were 9.8% and 10%, respectively, for assets X and Y. This additional measure is called the **coefficient of variation** (Coef_v) which considers the relative dispersion of data around the value expected. It is obtained by dividing the standard deviation (σ) by the expected return (k_e):

$$\boxed{\text{Coef}_v = \frac{\sigma}{k_e}}$$

So for assets X and Y, we get

$$\text{Coef}_v^x = \frac{\sigma_x}{k_e^x} = \frac{.0073}{.098} = .074$$

$$\text{Coef}_v^y = \frac{\sigma_y}{k_e^y} = \frac{.0353}{.10} = .353$$

The high coefficient of variation for asset Y (.353) confirms that it is more risky than asset X. Generally, the higher the coefficient of variation, the greater the risk associated with the asset. However, it is worthwhile to mention here that when we compare assets with that have an equivalent expected return, the test of coefficient of variation would not differ from the standard deviation test, but it does make a difference when the expected returns are different. In our example here, it just confirmed the standard deviation test, but it is not necessarily the case. Sometimes it would reverse the standard deviation case.

Example 1.2.1 Which of the two assets shown in Table E1.2.1 is riskier? Use both the standard deviation and coefficient of variation tests.

Based on the standard deviation, asset II has a higher standard deviation (5.5) than asset I (4.9) and therefore asset II is riskier. But based on the coefficient of variation test, asset I has a higher coefficient (1.63) than asset II (.46), so asset

TABLE E1.2.1

	Asset I	Asset II
Expected return	3%	12%
Standard deviation	4.9%	5.5%
Coefficient of variation	1.63	.46

I is riskier. Which test is more reliable? The coefficient of variation test is more reliable.

Long-Run Risk

In the long run, asset risk seems to be an increasing function of time. In other words, as time goes by, the variability of returns gets wider and the risk gets greater. That is why experience shows that the longer the life of an investment asset, the higher the risk involved in that asset. Figure 1.3 shows how the dispersion of return distribution of an asset gets wider and how the 1 standard deviation around the expected value gets larger over a 20-year period, assuming that the return expected stays the same.

3.1.3 RISK AVERSION AND RISK PREMIUM

Risk aversion is a general common behavior that refers to avoiding risky situations. Most investors are risk averse in the sense that, on average, they choose the less risky investment. This tendency comes with the understanding that the returns expected from the less risky investment would not be high. This also implies that if an investor chooses to invest in a higher risk asset, she or he would expect to collect a reward in terms of a high expected return and lower cost of investment. Generally speaking, we can say that a primary implication of the risk aversion dictates that the higher a security risk, the higher its expected return and the lower its price. Let's suppose that investment opportunity A is riskier than investment opportunity B, and let's suppose that both have the same share price of \$80 and the same expected return of \$8.00 per share. Since investors generally would choose the less risky investment, there would be a high demand for investment B and less demand for investment A. Over a reasonable amount of time, the higher demand on B and lower demand on A would increase the price of B and decrease the price of A. Let's assume that the price of B would rise to \$100 and the price of A would go down to \$60. Now the rates of return would no longer be 10% (8/80) for each. The rate of return for B would be $8/100 = 8\%$ and the rate for A would be $8/60 = 13.3\%$. The difference in the rates of return for the two investments, brought about by the change in prices due to changes in demand following the risk aversion, is called the **risk premium**. In this case it is $13.3\% - 8\% = 5.3\%$. This is to enforce the notion that riskier securities must attract and reward their investors with a higher expected return than that which is obtained by less risky securities.

3.1.4 RETURN AND RISK AT THE PORTFOLIO LEVEL

So far, the discussion has been on the return and risk of an individual asset or a single investment opportunity. In reality, it is uncommon to address a single asset in isolation. Financial assets and investment opportunities are often addressed in a

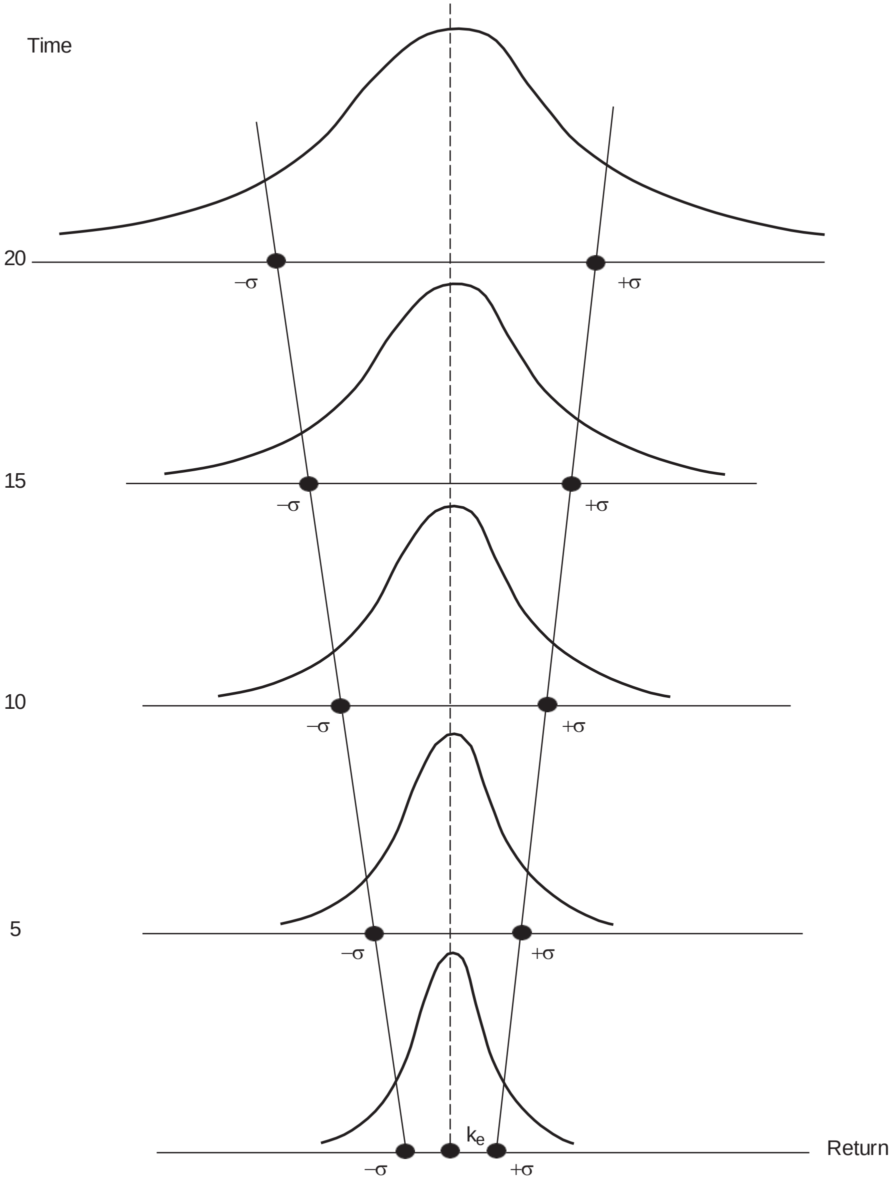

FIGURE 1.3

group and managed in a financial portfolio. Corporate investment, business funds, bank accounts, insurance and pension funds, and individual investments are all held in portfolios that are most likely to be diversified. Therefore, it would be more practical to talk about the return and risk of the entire portfolio as opposed to a single asset. The return or risk of a single component of the portfolio would become important only by its impact on the entire portfolio.

Portfolio Return

Portfolios are most likely to contain a number of individual assets, each in a different proportion. The market value of the entire portfolio (V_p) would be the summation of all values (V_i) of the individual assets.

$$V_p = V_1 + V_2 + \cdot \cdot \cdot + V_n$$

Each asset value would have its own proportion that would represent its own individual weight (w_i):

$$w_i = \frac{V_i}{V_p}$$

where

$$w_i = w_1, w_2, w_3, \ldots, w_n$$

and all individual weights would make up the entire weight of the portfolio:

$$w_1, w_2, w_3, \ldots, w_n = 100$$

In this same manner, the returns of all these individual assets (V_i) would constitute the return of the portfolio (r_p), and in a way that is commensurate to their weights.

$$r_p = r_1 + r_2 + \cdot \cdot \cdot + r_n$$

$$r_p = r_1 w_1 + r_2 w_2 + \cdot \cdot \cdot + r_n w_n$$

$$\boxed{r_p = \sum_{i=1}^{n} r_i w_i}$$

That is, the **portfolio return** is the summation of the weighted individual returns of its component assets, the weights being the proportions of those assets in the entire portfolio.

Example 1.4.1 An investor's portfolio contains two different stocks, two different bonds, and two different mutual funds, with the proportions and returns shown in Table E1.4.1. Calculate the portfolio return.

The portfolio return is 9.26%.

There is another aggregate way to calculate the portfolio return based on the change in its entire market value within a certain period of time. The ratio of

TABLE E1.4.1

Asset	r_i (%)	w_i (%)	$r_i w_i$
Stock I	13.5	23	.0311
Stock II	12	18	.0216
Bond I	6.5	15	.00975
Bond II	6	15	.009
MF I	7.5	17	.01275
MF II	7	12	.0084

Portfolio return = $\sum_{i=1}^{6}$ □□□□.0926

the change in value to the original value would estimate the rate of return. In other words, the portfolio rate of return, here, is just the percentage change in the value of the entire portfolio between two points in time:

$$\boxed{r_p = \frac{V_p^2 - V_p^1}{V_p^1}}$$

where V_p^1 is the market value of the entire portfolio at the start of the year, and V_p^2 is the market value of the portfolio at the end of the year.

Example 1.4.2 Suppose that the investment portfolio for a small business is estimated at one point by \$425,000, and its market value went to \$538,620 one year later. That is, $V_p^1 = \$425{,}000$; $V_p^2 = \$538{,}620$. What is the portfolio rate of return?

$$r_p = \frac{V_p^2 - V_p^1}{V_p^1}$$

$$r_p = \frac{\$538{,}620 - \$425{,}000}{\$425{,}000}$$

$$r_p = \frac{\$113{,}620}{\$425{,}000} = 26.7\%$$

The portfolio expected rate of return (k_p) would be calculated in the same manner as the portfolio rate of return: that is, as the summation of the weighted individual expected returns (k_{ei}):

$$\boxed{k_p = \sum_{i=1}^{n} k_{ei} w_{ei}}$$

Portfolio Risk

Unlike the portfolio return, portfolio risk is not a weighted average of the individual risk of components. **Portfolio risk** can be reduced by adding more and different assets to the portfolio. In other words, the more diversified the portfolio, the less the total portfolio risk. More important is the degree of correlation among the individual assets in a portfolio. The less correlated the assets, the less the portfolio risk. In this sense, diversification would not be as effective in reducing risk unless it involves either negatively correlated assets, or at least the lowest positively correlated assets. The correlation coefficient (Corr) can measure how two variables move against each other. The Corr value ranges from -1, referring to a perfectly negative correlation when variables move in opposite directions, to 1, a perfectly positive correlation where variables move in the same direction. If diversification includes the assets that are negatively correlated, they would move opposite each other and cancel each other out, resulting in risk reduction. However, if diversification brings assets that are strongly positively correlated, risk cannot be diversified away. Let's take a look at the following examples and reflect on how the positively and negatively correlated asset returns affect the portfolio risk.

Example 1.4.3 Let's take a look at two pairs of assets, X and Y and Z and W. Table E1.4.3a shows the returns for assets X and Y where the variances of the two sets of return were calculated individually in columns (7) and (12) as

$$\sigma_X^2 = \sum_{i=1}^{5} (k_i^x - k_e^x)^2 \cdot \Pr_i = .046$$

and

The standard deviations of the two sets of return were calculated as

$$\begin{aligned} \sigma_x &= \sqrt{\sum_{i=1}^{5} (k_i^x - k_e^x)^2 \cdot \Pr_i} \\ &= \sqrt{.046} = .214 \end{aligned}$$

$$\begin{aligned} \sigma_y &= \sqrt{\sum_{i=1}^{5} (k_i^y - k_e^y)^2 \cdot \Pr_i} \\ &= \sqrt{.067} = .259 \end{aligned}$$

TABLE E1.4.3a Returns for Assets X and Y

		Asset X					Asset Y					
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
	Pr_i	k_i^x	k_e^x	$k_i^x - k_e^x$	$(k_i^x - k_e^x)^2$	$(k_i^x - k_e^x)^2 \cdot Pr_i$	k_i^y	k_e^y	$k_i^y - k_e^y$	$(k_i^y - k_e^y)^2$	$(k_i^y - k_e^y)^2 \cdot Pr_i$	$(k_i^x - k_e^x)(k_i^y - k_e^y) \cdot Pr_i$
			$\frac{\sum k_i^x}{5}$	(3)−(4)	$(5)^2$	(6) (2)		$\frac{\sum k_e^y}{5}$	(8)−(9)	$(10)^2$	(11) (2)	(5) (10) (2)
k_1	.20	−.12	.155	−.275	.0756	.015	−.15	.174	−.324	.105	.021	.0178
k_2	.15	.405	.155	.25	.0625	.0094	.50	.174	.326	.1063	.016	.0122
k_3	.25	−.07	.155	−.225	.0506	.0126	−.08	.174	−.254	.0645	.016	.0143
k_4	.18	.38	.155	.225	.0506	.0091	.45	.174	.276	.0762	.014	.0112
k_5	.22	.18	.155	.025	.000625	.00014	.15	.174	−.024	.00058	.00013	−.00013
						.046					.067	.0554

The covariance between the two sets of return, Cov(x, y), is calculated in column (13) as

$$\text{Cov}(x, y) = \sum_{i=1}^{5} (k^x_i - k^x_e)(k^y_i - k^y_e) \cdot \text{Pr}_i$$

$$= .0554$$

and finally, the correlation coefficient (Corr) between the two sets of return is calculated as

$$\text{Corr}_{x,y} = \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y}$$

$$= \frac{.0554}{.214(.259)}$$

$$= 99.9\%$$

A correlation coefficient this close to $+100$ would be considered an indication of perfectly positively correlated assets which exhibit similar dynamics, which makes them move up and down in tandem. This type of matching pattern would not benefit from diversification in risk reduction. Figure E1.4.3a shows such a synchronized movement in the returns of those assets.

The second set of assets, Z and W, is presented in Table E1.4.3b, and the same parameters are calculated in the same manner as that in which they were calculated in Table E1.4.3a. The variances of the assets are at columns (7) and (12):

$$\sigma_z^2 = \sum_{i=1}^{5} (k^Z_i - k^Z_e)^2 \cdot \text{Pr}_i$$

$$= .0339$$

$$\sigma_w^2 = \sum_{i=1}^{5} (k^w_i - k^w_e)^2 \cdot \text{Pr}_i$$

$$= .0129$$

and the standard deviations are

$$\sigma_z = \sqrt{.0339} = .1841$$

$$\sigma_w = \sqrt{.0129} = .1136$$

The covariance is calculated in column (13) as

$$\text{Cov}(z, w) = \sum_{i=1}^{5} (k^z_i - k^z_e)(k^w_i - k^w_e)^2 \cdot \text{Pr}_i$$

$$= -.020353$$

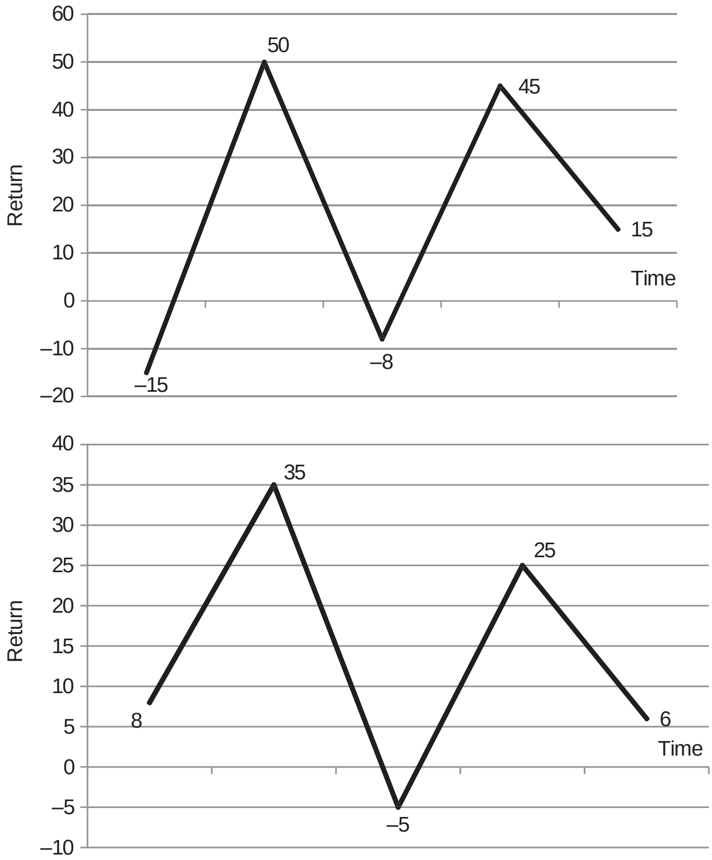

FIGURE E1.4.3a

The correlation coefficient (Corr) is

$$\mathrm{Corr}_{z,w} = \frac{\mathrm{Cov}(z, w)}{\sigma_x \sigma_w}$$

$$= \frac{-.020353}{.1841(.1136)} = 97.3\%$$

A correlation coefficient of -97.3% shows the case opposite to that of the X and Y combination. It indicates that assets Z and W are almost perfectly negatively correlated, which means that the return changes of these assets go up and down opposite to each other. This is the ideal opportunity for these assets to cancel each other out. If one is down, the other is up to compensate — that is the beauty of diversification. The combination of such assets in a portfolio gives the opportunity to have an optimal impact of diversifying the risk away. Figure E1.4.3b shows how the return patterns act opposite each other in a consistently contrasting manner.

TABLE E1.4.3b Returns for Assets Z and W

		Asset Z					Asset W					
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
	Pr_i	k_i^z	k_e^z	$k_i^z - k_e^z$	$(k_i^z - k_e^z)^2$	$(k_i^z - k_e^z)^2 \cdot Pr_i$	k_i^w	k_e^w	$k_i^w - k_e^w$	$(k_i^w - k_e^w)^2$	$(k_i^w - k_e^w)^2 \cdot Pr_i$	$(k_i^z - k_e^z)(k_i^w - k_e^w) \cdot Pr_i$
k_1	.20	.40	.20	.20	.04	.008	.08	.138	−.058	.0034	.00068	−.00232
k_2	.15	.10	.20	−.10	.01	.0015	.35	.138	.212	.045	.00675	−.00318
k_3	.25	.38	.20	.18	.0324	.0081	−.05	.138	−.188	.0077	.001925	−.00846
[illegible]	[illegible]	−.10	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
k_5	.22	.22	.20	.02	.0004	.000088	.06	.138	−.078	.0061	.001342	−.000343
						.0339					.0129	−.020353

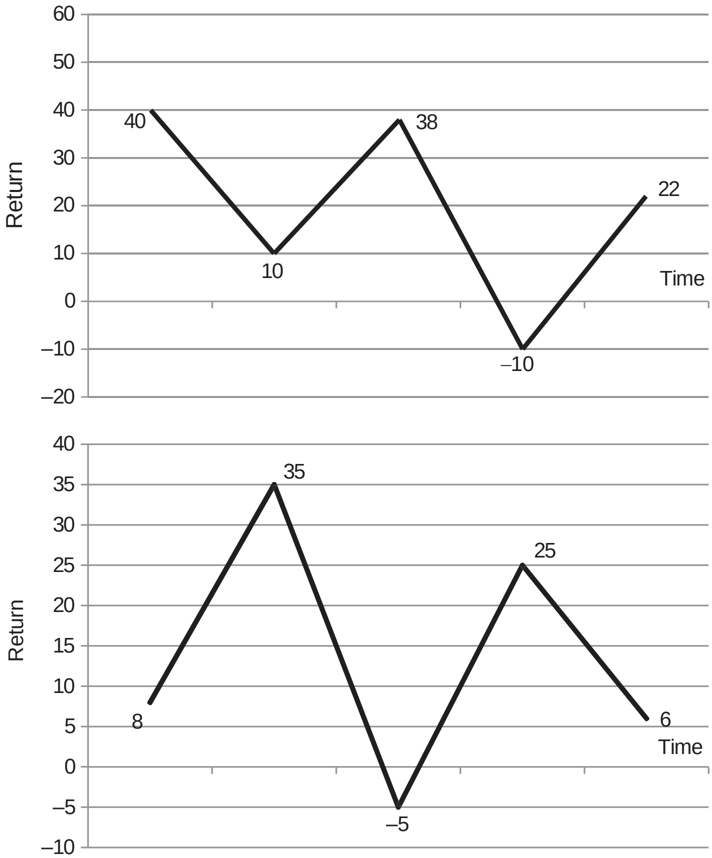

FIGURE E1.4.3b

Combining assets into portfolios would probably reduce the risk even for those assets that are positively correlated. In Tables E1.4.3c and E1.4.3d we combined assets X and Y and obtained an average vector of returns for the combination XY. We also combined assets Z and W and obtained an average vector of returns for the combination ZW. The standard deviation test shows that the combination helps reduce risk even for combining X and Y, which are perfectly positively correlated, as we have seen. The standard deviation of the combined set XY ($\sigma_{xy} = .195$) is still less than either of the assets taken individually, where $\sigma_x = .214$ and $\sigma_y = .259$. This means that the combined assets showed that it is not as risky as either of the individual assets alone. This standard deviation test shows that much better results can be obtained when we combine the negatively correlated assets Z and W. The standard deviation of the combined set ZW ($\sigma_{zw} = .056$), is much less than either of the assets' standard deviation, where $\sigma_z = .1841$ and $\sigma_w = .1136$. It is a further proof that combining assets into portfolios would increase diversification and reduce risk. However,

TABLE E1.4.3c Combined Assets X and Y with Average Returns

Return	Pr_i	k_i^{xy}	k_e^{xy}	$k_i^{xy} - k_e^{xy}$	$(k_i^{xy} - k_e^{xy})^2$	$(k_i^{xy} - k_e^{xy})^2 \cdot Pr_i$
k_1	.20	−.135	.164	.029	.00084	.00017
k_2	.15	.452	.164	.288	.083	.0124
k_3	.25	−.075	.164	−.239	.057	.0142
k_4	.18	.415	.164	.251	.063	.0113
k_5	.22	.165	.164	.001	.000001	.00000022
					$\sum_{i=1}^{5} (k_i^{xy} - k_e^{xy})^2 \cdot Pr_i$	.038

TABLE E1.4.3d Combined Assets Z and W with Average Returns

Return	Pr_i	k_i^{zw}	k_e^{zw}	$k_i^{zw} - k_e^{zw}$	$(k_i^{zw} - k_e^{zw})^2$	$(k_i^{zw} - k_e^{zw})^2 \cdot Pr_i$
k_1	.20	.24	.169	.071	.00504	.001
k_2	.15	.225	.169	.056	.00314	.00047
k_3	.25	.165	.169	−.004	.000016	.000004
k_4	.18	.075	.169	−.094	.00884	.00159
k_5	.22	.14	.169	−.029	.00084	.00018
					$\sum_{i=1}^{5} (k_i^{zw} - k_e^{zw})^2 \cdot Pr_i$.	0032

the extent of risk reduction depends primarily on the degree and sign of the correlation between assets. In reality, most of the assets are positively correlated. Studies show that on average, randomly selected assets shows a correlation coefficient around .60. The lower the positive correlation, the better the results of the combination.

In another abstract presentation, Figure 1.4 shows three possible ways to combine two assets in a portfolio: two extreme combinations and one common combination.The assets are A with an expected return of k_A and a risk level of σ_A, and B with an expected higher return of k_B and a higher risk level σ_B.

- The first extreme case of combination occurs at any point along the straight line AB if assets A and B are perfectly positively correlated. This combination cannot benefit much from diversification.
- The second extreme case of combination occurs at any point along BCA where a zero risk can be achieved with a rate of return equal to k_c when the allocation of the two assets can be achieved in reverse proportion to their risk levels. This combination is the show case for the benefit of diversification.
- The third case of combination occurs at any point along curve BA. It is the most likely to occur because assets are often neither negatively nor positively perfectly correlated. The correlation would often be on a moderate level, and

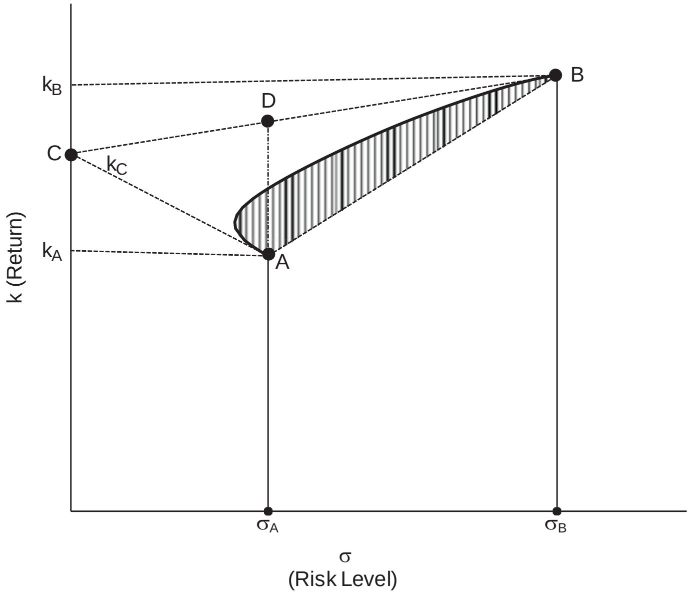

FIGURE 1.4

the combination of assets can enjoy a wide range of returns, from k_A to k_B, for a wide range of risk levels, from σ_A to σ_B. The curve would include all the possible combinations that are better alternatives to any point along straight line *AB* but lower alternatives to most of the points along *BCA*, which would offer higher rates of return for the same level of risk, especially along segment *BD*.

3.1.5 MARKOWITZ'S TWO-ASSET PORTFOLIO

A great deal of what has been known about risk and return and portfolio construction traces back to a pioneering study by Harry Markowitz published in 1952 (*Journal of Finance*, 7, pp. 77 – 91). A major issue that was addressed is portfolio diversification of assets and the positive outcome on the portfolio return through the compensatory effect of assets that move in different directions. A well-known fundamental example illustrates the effect of combining two individual assets of different return and risk rates on the portfolio return and risk.

Figure 1.5 shows what happens if an investor decides to invest in two different choices of stocks: stock I (SI), with an expected return of 8% and a low risk (represented by the standard deviation of return) of 15%, and stock II (SII), which offers a higher return of 12% but at a higher risk of 22%. The logical expectation is to calculate the combined return and risk for the mix if we know how much investment an investor is willing to dedicate to each stock. Let's assume that this investor or this portfolio manager is willing to dedicate 55% of investment to

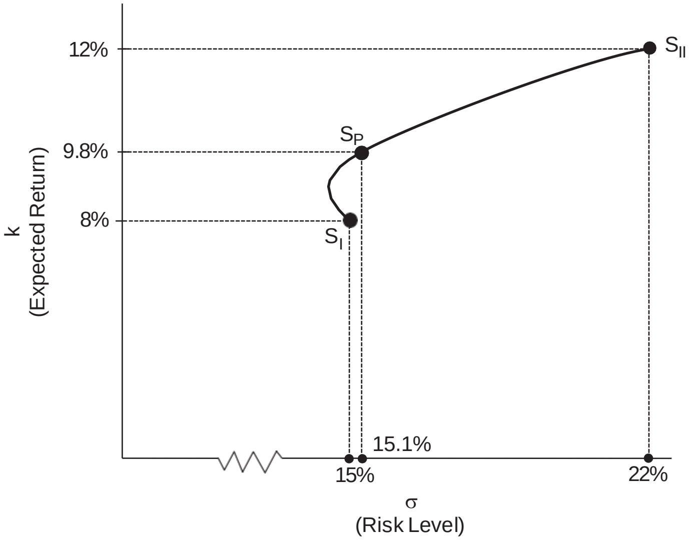

FIGURE 1.5

stock I and 45% to stock II. The portfolio rate of return would be calculated as the weighted average of two returns:

$$
\begin{aligned}
k_p &= w_1k_1 + w_2k_2 \\
&= .55(.08) + .45(.12) \\
&= 9.8\%
\end{aligned}
$$

As for the portfolio risk, it would be determined by the standard deviation of the combined assets given a correlation between the two assets of .38.

$$\sigma_{I,II} = \sqrt{\sigma_I^2 w_I^2 + \sigma_{II}^2 w_{II}^2 + 2\mathrm{Corr}_{I,II}(w_I\sigma_I)(w_{II}\sigma_{II})}$$

$$= \sqrt{(.15)^2(.55)^2 + (.22)^2(.45)^2 + 2(.38)(.55)(.15)(.45)(.22)}$$

$$= \sqrt{.0228} = 15.1\ \%$$

So the risk level of the combined stocks in an asset is less than the weighted average risk of the two individual assets, which would have been:

$$(.55)(.15) + (.45)(.22) = 18.2\%$$

Therefore, a combination of assets at S_p would yield a 9.8% rate of return at a reasonable level of risk (15.1%). If we move from this hypothetical example of only two assets in a portfolio to the reality of the investment in the market,

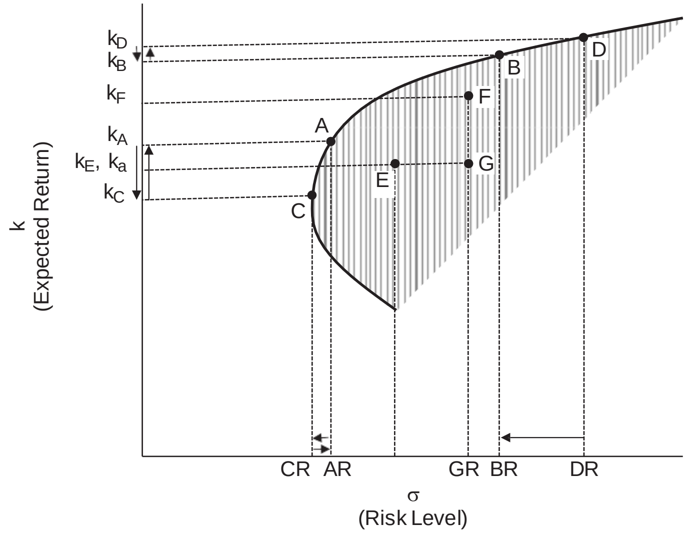

FIGURE 1.6

we would find that there are a large number of assets forming a large number of combinations and producing a large number of portfolios. The broken-egg-shaped area in Figure 1.6 represents all the combinations of assets that are attainable to all investors with their different objectives and different risk and return preferences.

Following are some major observations on Figure 1.6:

- The shaded broken-egg-shaped area is a locus of portfolios with all possible combinations of assets representing a wide range of investor preferences regarding risk and return.
- The solid curve represents the diversified portfolios with the highest returns for any given risk level between CR and DR. Markowitz called this the **efficient portfolio curve**. It is also called the **frontier of risky portfolios**.
- Point *D* is the portfolio that yields the highest return (k_D) but bears the highest level of risk (DR).
- Point *C* is the portfolio that yields the lowest return (k_C) but enjoys the lowest level of risk (CR).
- Segment *DB* contains a collection of portfolios that enjoy a trade-off between risk and return in favor of the risk side. For example, moving from *D* to *B* means getting a slightly lower return than k_D but for a greater reduction in the risk level, from DR to BR. Similarly, moving from *B* to *D* means gaining a slightly higher return than k_B but carrying more risk from BR to DR.
- Segment *AC* contains a collection of portfolios that enjoy a trade-off between risk and return in favor of the return. For example, moving from

A to C means accepting a greater reduction in return, from k_A to k_C for a lower reduction in risk, from AR to CR. Similarly, moving from C to A means getting a much higher return for accepting a little more risk, from CR to AR.

- Segment AB contains all the portfolios that exhibit an almost equal trade-off between risk and return. In other words, gaining or losing a certain amount of return comes with gaining or losing a compatible amount of risk.
- Inside the shape we can observe that moving toward the northeast means getting portfolios with higher return and higher risk. On the contrary, moving toward the southwest means getting portfolios with lower return and lower risk.
- Portfolio F is preferred to portfolio G because it yields more return for the same amount of risk.
- Portfolio E is preferred to portfolio G because it enjoys a much lower level of risk for the same rate of return.

3.1.6 LENDING AND BORROWING AT A RISK-FREE RATE OF RETURN

Looking at Figure 1.7, let's assume that an investor wants to split his initial investment between asset A on the efficient portfolio curve and U.S. Treasury bills, which offer a risk-free rate of return of 5%. Suppose that A yields 12% at

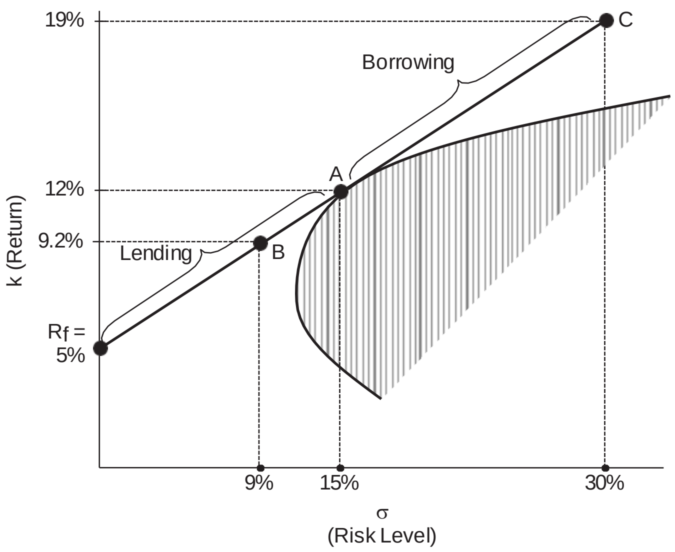

FIGURE 1.7

a risk level of 15%. The investor would like to have 60% of his money invested in asset A and 40% invested in Treasury bills.

The investor in this case is lending 40% of his money to Treasury bills. His rate of return would be

$$.40(.05) + .60(.12) = 9.2\%$$

and his level of risk would be

$$.40(0) + .60(.15) = 9\%$$

He would be at point B. This means that he could be at any point along the line AR_f, depending on the proportions of his investment between asset A and Treasury bills.

Now, let's assume that he borrows at the risk-free rate of 5% an amount of money equal to his own money and invests the total (his own and the borrowed money) in asset A alone. His return would be

$$2(.12) - .05 = 19\%$$

and his risk would be

$$2(.15) + .05(0) = 30\%$$

He would be at point C, which means that he could be at any point along CA depending on how much he borrows and how much risk he tolerates.

3.1.7 TYPES OF RISK

The risk of the individual assets and of the portfolios that we talked about so far, is the type of **diversifiable risk**, which can be reduced by diversification of assets within portfolios. This type is usually firm-specific risk. It is also called **unsystematic risk**, for it is related to internal conditions and circumstances and varies from firm to firm. It is due to a random set of specifications unique to a specific firm, such as lawsuits in which the firm is involved, the marketing program it conducts, a workers' strike that it faces, or the type and quality of contracts it wins or loses. All these events and circumstances can be mitigated with a certain degree of the firm's diversification of assets. The other type of risk is the **undiversifiable** or **systematic risk**, which is general and market related. It is due to circumstances and conditions that all firms are affected by simultaneously and with no discrimination. The state of the economy is the classic example, highlighted by an impact such as inflation, recession, unemployment, interest rate fluctuation, war, severe weather, or political unrest. Nothing can be done to eliminate or even reduce the risk stemming from such external factors,

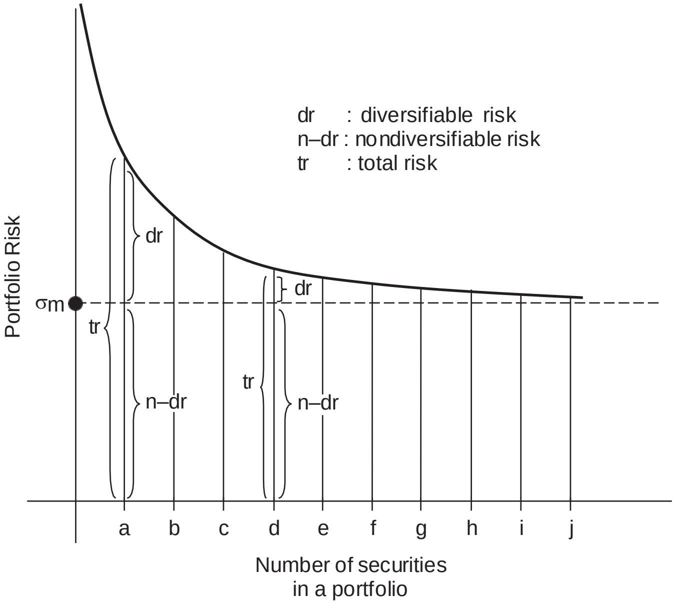

FIGURE 1.8

nor can it be avoided by diversification of assets. It can, however, be assessed by monitoring how a particular asset tends to respond to market circumstances and changes, and can be addressed by an analysis of the capital asset pricing model and measured by the financial beta. Figure 1.8 shows that as a financial portfolio contains more and more individual assets and securities, portfolio risk tends to decline until asymptotically it approaches the systematic nondiversifiable risk. At any number of individual securities in the portfolio, such as a or d, we can see how the diversifiable and nondiversifiable risks are distributed.

3.2 The Capital Asset Pricing Model (CAPM)

The capital asset pricing model (CAPM) is a technical tool used to analyze the relationship between a financial asset's expected return and the nondiversifiable market risk. A major component in this model is the financial beta (β), which we hinted to before but will address here in more detail.

3.2.1 THE FINANCIAL BETA (β)

Beta (β) is a mathematical tool to measure systematic undiversifiable market risk. It is, in this sense, an index of the extent to which a security return moves in response to changes in the overall market. This would make it as a measure of securities volatility, in relation to an average security, represented by the state of the market. Market return is an aggregate measure of the return of all securities traded in a market at a specific time. The beta value can be positive or negative and generally ranges between –2.5 and 2.5. A value of 1.00 denotes the full impact of market risk. Any individual security with a beta of 1.00 indicates that the return pattern of that security moves up and down perfectly with the market return. A value of zero refers to total independence from market impact. A value of more than 1.00, such as 2.00, reveals that a security is twice as volatile as the average security in the market. A negative value says that the asset return pattern moves in the opposite direction from the market. Table 2.1 shows beta coefficients of selected American companies estimated at some point in time. The estimation changes for the same company from time to time.

Mathematically, beta is obtained by dividing the covariance between the individual security return (k_i) and the market return (k_m) by the variance of market return.

$$\boxed{\beta = \frac{\mathrm{cov}(k_i, k_m)}{\mathrm{Var}(k_m)}}$$

In this sense, beta is a concept of correlation to assess how one security return is correlated with the rest in the market. From another perspective, beta measures the percentage change in one security return as it responds to changes in the external market. It can therefore be interpreted as the financial elasticity of the

TABLE 2.1 Beta Estimates for Selected American Companies

Company	Beta
AOL	2.46
Dell	2.23
Microsoft	1.82
Texas Instrument	1.75
Intel	1.70
GE	1.16
GM	1.10
Colgate-Palmolive	1.03
Family Dollar Store	.99
K-Mart	.98
ATT	.98
McGraw-Hill	.81
Gillette	.76
MY Times	.71
JC Penney	.52
Johnson & Johnson	.49
Campbell	.41
Exxon	.36

change in a given asset relative to market change. Accordingly, beta becomes the slope of the regression line between the changes in market return and the corresponding response of the asset return. In figure 2.1 the changes in market return are tracked down on the horizontal axis, and the responses to them by a given security are tracked down on the vertical axis. The result would be the regression line $y = \alpha + \beta x + e$, with its slope standing for beta, calculated by dividing the change in the vertical axis over the change in the horizontal axis:

$$\text{beta} = \beta = \frac{.6y}{.6x}$$

If we track down the change in the market rate when it is increased from 7.3 to 9.3, the linear equation line of $y = -8 + l.5x$ would allow the return of the asset k_i to increase from 3 to 6. Therefore, we can obtain the slope of the line:

$$\text{slope} = \frac{.6y}{.6x} = \frac{6-3}{9.3-7.3} = \frac{3}{2} = 1.5$$

which is the value of β in the equation of the line. This says that the asset rate of return follows the market return, but even more robustly — its volatility is one and a half times as much as that of the volatility of the market return. For example, if the market rate increases by 5%, this asset's rate would increase by 7.5%.

We can also calculate the beta value by the formula method.

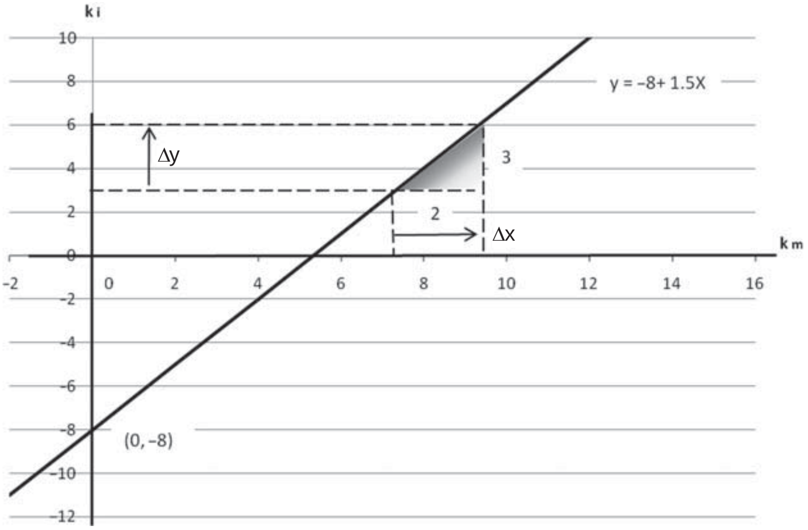

FIGURE 2.1

Example 2.1.1 We can calculate beta for corporation X given 10 periodic rates of return (k_i^x) and market rates for the same period (k_i^m). In Table E2.1.1 we calculate the expected return for both as the averages k_e^x and k_e^m, and we proceed to calculate the covariance between the two sets of rates and the variance of the market. Beta would be calculated by dividing the covariance by the market variance.

$$\text{Cov}(x, m) = \frac{\sum_{i=1}^{10} (k_i^x - k_e^x)(k_i^m - k_e^m)}{N}$$

$$= \frac{.04773}{10} = .0048$$

$$\text{Var}(m) = \frac{\sum_{i=1}^{10} (k_i^m - k_e^m)^2}{N}$$

$$= \frac{.0348}{10} = .0035$$

$$\beta_x = \frac{\text{Cov}(x, m)}{\text{Var}(m)}$$

$$= \frac{.0048}{.0035} = 1.37$$

Note that β_x is just the beta of asset X. The beta for the portfolio would be equal to the weighted average of betas for all individual assets within the portfolio:

$$\beta_p = \sum_{i=1}^{n} \square\square\square\square$$

TABLE E2.1.1 Rates of Return for Corporation x and the Market for 10 Periods

	Corporation x			Market				
Period	k_i^x	k_e^x	$k_i^x - k_e^x$	k_i^m	k_e^m	$k_i^m - k_e^m$	$(k_i^m - k_e^m)^2$	$(k_i^x - k_e^x)(k_i^m - k_e^m)$
1	−.06	.105	−.165	.027	.082	−.055	.003	.0091
2	.27	.105	.165	.095	.082	.013	.00017	.0021
1	.065	.105	−.04	.038	.082	−.044	.0019	.0018
2	.13	.105	.025	.055	.082	−.027	.00073	−.00067
3	.055	.105	−.05	−.017	.082	−.099	.0098	.0049
4	.28	.105	.175	.176	.082	.094	.0088	.0164
5	−.045	.105	−.15	.119	.082	.037	.0014	−.0055
6	.03	.105	−.075	.128	.082	.046	.0021	−.0034
7	.35	.105	.245	.156	.082	.074	.0055	.0181
10	−.025	.105	−.13	.044	.082	−.038	.0014	.0049
							.0348	.04773

where β_p is the portfolio beta, β_i is the beta for any individual asset within the portfolio, and w_i is the proportion of asset i in the entire portfolio, which contains n assets.

3.2.2. THE CAPM EQUATION

Now that we know what beta is, we can write the central equation of the capital asset pricing model, where beta is an essential factor to calculate the required rate of return on any asset (k_i) given the asset's beta (β_i), the market's required rate of return (k_m), and the riskless rate of return, which is traditionally the rate of return on a U.S. Treasury bond (R_f):

$$\boxed{k_i = R_f + \beta_i(k_m - R_f)}$$

In this model the required rate of return for any asset is obtained by adding the risk-free rate of return to the market risk premium, given that this premium is:

1. The difference between the market required rate of return and the risk-free rate of return: $k_m - R_f$
2. Adjusted to that asset's index of risk by being multiplied by beta: $\beta_i(k_m - R_f)$

Example 2.2.1 What would be the required rate of return for corporation Y, which has a beta of 1.85, given that the return on the market portfolio of assets is 12% and the risk-free rate is 6.5%?

$$
\begin{aligned}
k_i &= R_f + \beta_i(k_m - R_f) \\
&= .065 + 1.85(.12 - .065) \\
&= 16.67\%
\end{aligned}
$$

So the market risk premium is 5.5% (.12 – .065), and it went to a little more than 10% when it was adjusted to the asset's index of risk, a beta of 1.85. When the result of the adjustment was added to the risk-free rate of 6.5%, we got the corporation required rate of 16.67%.

Algebraically, we can obtain any of β_i, R_f, and k_m if the other variables in the equation are available. To find beta:

$$
\begin{aligned}
k_i &= R_f + \beta_i(k_m - R_f) \\
k_i - R_f &= \beta_i(k_m - R_f)
\end{aligned}
$$

$$
\boxed{\beta_i = \frac{k_i - R_f}{k_m - R_f}}
$$

$$
\begin{aligned}
\beta_i &= \frac{.1667 - .065}{.12 - .065} \\
&= 1.85
\end{aligned}
$$

To find the free risk of return (R_f):

$$
\begin{aligned}
k_i &= R_f + \beta_i(k_m - R_f) \\
k_i &= R_f - \beta_i R_f + \beta_i k_m \\
k_i - \beta_i k_m &= R_f(1 - \beta_i)
\end{aligned}
$$

$$
\boxed{R_f = \frac{k_i - \beta_i k_m}{1 - \beta_i}}
$$

$$
\begin{aligned}
&= \frac{.1667 - 1.85(.12)}{1 - 1.85} \\
&= \frac{-.0553}{-.85} \\
&= .065
\end{aligned}
$$

To find the market rate of return (k_m):

$$k_i = \mathrm{R_f} + \beta_i(k_m - R_f)$$

$$k_i = \mathrm{R_f} + \beta_i k_m - \beta_i R_f$$

$$k_i + R_f(\beta_i - 1) = \beta_i k_m$$

$$\boxed{k_m = \frac{k_i + R_f(\beta_i - 1)}{\beta_i}}$$

$$= \frac{.1667 + .065(1.85 - 1)}{1.85}$$

$$= .12$$

3.2.3 THE SECURITY MARKET LINE

When we graph the CAPM equation, we get a straight line with a positive slope equal to beta (1.85 in the last example). This line is called the **security market line** (SML). From Example 2.2.1 we have all the points we need to draw the SML. As shown in Figure 2.2, the level of risk, as measured by beta, is on the horizontal axis, and the required rates of return are on the vertical axis. The risk-free rate of 6.5% is associated with zero beta, the market rate of 12% is associated with a beta value of 1.00, and the return k_i of 16.67% is associated with a beta of 1.85. The vertical line BD represents the market risk premium, which is obtained as $BE - DE$, standing for $k_m - R_f$ in the equation. All assets that have betas higher than 1.00 would have a risk premium higher than the market risk premium. Our beta is 1.85, which makes the risk premium much higher than the market risk premium, as depicted by the vertical line CE, which is $\mathrm{CF} - \mathrm{EF} = 16.67 - 6.5 = 10.17$. Similarly, all assets that have betas below 1.00 will have risk premiums below the market risk premium: For example, firm S, with a beta of. 5. The risk premium for this firm would be represented by the line GH, which is $GI - HI$, and is lower than the market risk premium.

The slope of SML stands for the degree of risk aversion. A steeper SML would reflect a higher degree of risk aversion, and a flatter SML would reflect a lower degree of risk aversion in the economy. Also, the steeper the SML, the higher the risk premium and the higher the required rate of return on the risky assets with higher betas.

SML Shift by Inflation

The risk-free rate of return (R_f) is considered the price of money to a riskless borrower. It is therefore, like any other price, affected by inflation. In fact, it includes a built-in component designed to absorb the impact of inflation. This

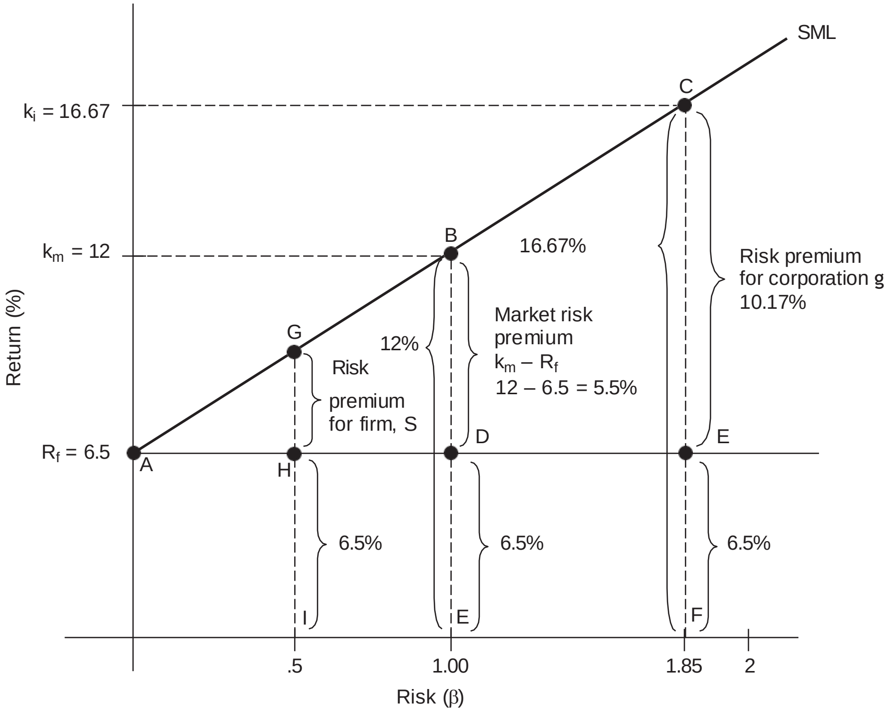

FIGURE 2.2

component, called the **inflation premium** (IP), is to protect the investor's purchasing power from declining as prices rise. The second component is the real core rate (k^0), which is inflation free.

$$\boxed{Rf = k^0 + \text{IP}}$$

Let's assume that the risk-free rate of Example 2.2.1, which is 6.5%, is in fact a combination of the real rate k^0, which is 2.5%, and the inflation premium 4%.

Since the SML originates from the R_f point, any increase in inflation would lead to an increase in the IP component and to R_f, and it would cause the point of SML origin to go up, shifting the entire line up. Figure 2.3 shows how the SML shifts to a higher position (SML_2), originating from the 8% rate if the inflation rises by 1.5 points, from 4% to 5.5%.

$$R_f^1 = k^0 + \text{IP}_1$$

$$= 2.5 + 4 = 6.5$$

$$R_f^2 = k^0 + \text{IP}_2$$

$$= 2.5 + 5.5 = 8.00$$

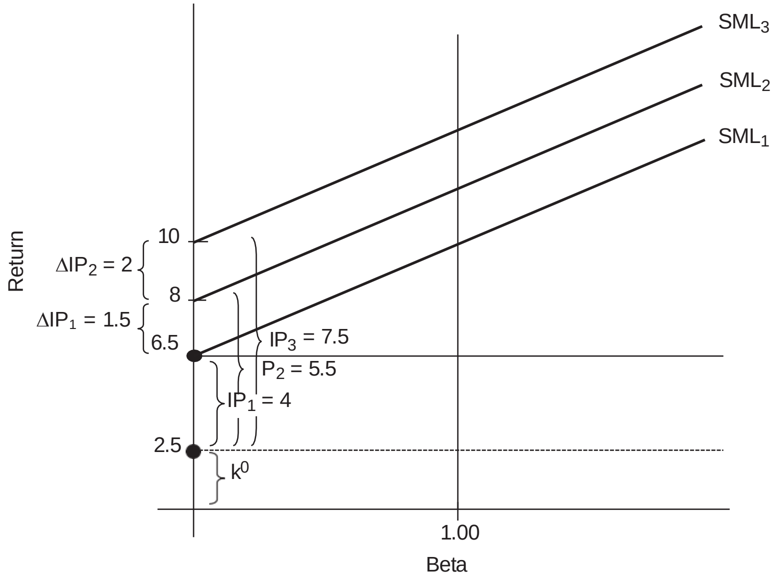

FIGURE 2.3

and if inflation continues to rise to, say, 7.5, the SML shifts again to SML_3, originating from $R_f = 10$.

$$R_f^3 = k^0 + \text{IP}_3$$
$$= 2.5 + 7.5 = 10.0$$

3.2.4 SML SWING BY RISK AVERSION

The slope of the security market line (SML) represents how much risk aversion investors usually exhibit. The line would therefore swing up and down to reflect the change in investor's risk aversion. The slope would be equal to zero and the SML would be horizontal at the risk-free level of return (R_f), and there would be no risk premium to the point where even risky assets would sell at the risk-free level of return. As the risk aversion starts to rise and the risk premium starts to grow, the SML would pivot at the R_f point and its other end would start to rise according to how much risk premium there is. From that point, the line would continue to swing up. Figure 2.4 shows that when risk aversion rises, the market risk premium (MRP) would rise — in this example from 4% to 9% (vertical *FG* to *EG*)— and consequently, the market required rate (k_m) would jump from 10% to 15% (vertical *FK* to *EK*), and our risky asset return (k_i) of the 1.75 beta would be up from 13% to 21.75%. This asset risk premium (RAP) would shoot

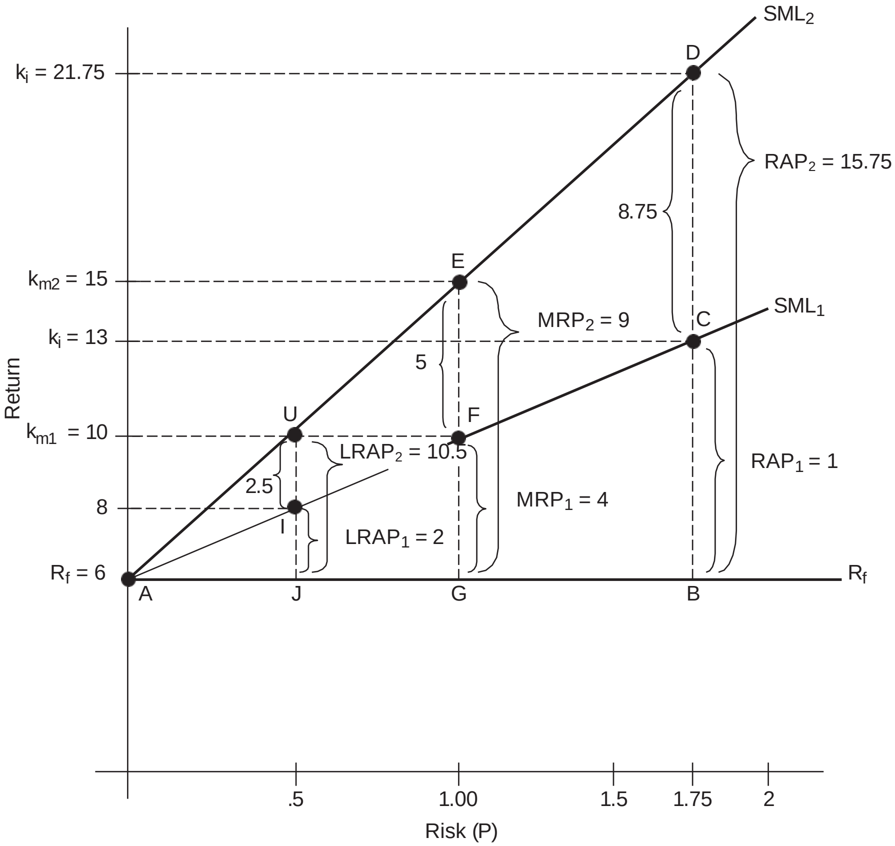

FIGURE 2.4

from 7% to 15.75% (vertical *CB* to *DB*).

$$k_i^1 = R_f + \beta_i(k_m^1 - R_f)$$

$$= .06 + 1.75(.10 - .06)$$

$$= .06 + .07$$

$$= .13$$

where

$$k_i^2 = R_f + \beta_i(k_m^2 - R_f)$$

$$= .06 + 1.75(.15 - .06)$$

$$= .06 + .1575$$

$$= .2175$$

where $1.75(.10 - .06) = 7\%$ and $1.75(.15 - .06) = 15.75\%$ are the risky asset's risk premiums during the change. This change would result in increasing the slope of the line and making it steeper. SML_1 would swing up to SML_2. It is clear that the impact of the change in the risk aversion level would be more pronounced on assets that are riskier, with a higher beta, compared to less risky assets that have betas below 1.00. This is shown by the increase in the required rate of return between the two types of assets for the risky asset with 1.75 beta, the required rate of return increased by 8.75% (from 13% to 21.75%), while the required rate of return for the asset of .5 beta rose by 2.5% (from 8% to 10.5%). This means that such a risky asset, with a 1.75 beta, faced an increase in the required return three and a half times higher than what the less risky asset of .5 beta faced as an increase in its required return.

A last word regarding the slope of SML is worth emphasizing. The SML slope is not equal to beta, as may be guessed by looking at the equation of the model:

$$k_i = R_f + \beta_i(k_m - R_f)$$

As we have seen before, beta is equal to the slope of the regression line that describes the response of the return of a certain asset to a change in the market return. Let's consider the difference among three assets as they respond to a certain change in the market return (see Figure 2.5). Let's assume that the market return at some point is 8% and that it goes up to 12%. That is an increase of 50%. Let's also assume that our three assets, X, Y, and Z, all have 8% rates and that they respond to such a change in the market return in the following manner:

- *Asset X*: The return increases by the same percentage as the market, 50%. Its return of 8% goes to 12%.
- *Asset Y*: The return increases by 100%. Its return goes from 8% to 16%.
- *Asset Z*: The return increases by 25%. Its return goes from 8% to 10%.

Asset X would have a 45° line going through the original point. Its slope is equal to 1 because the change in the asset return is the same as the change in the market return:

$$\text{slope}_x = \frac{\text{rise}}{\text{run}} = \frac{.6.\ \text{asset return}}{.6.\ \text{market return}}$$
$$= \frac{12 - 8}{12 - 8} = 1 = \text{beta}$$

The line equation would be

$$Y = a + b_x$$
$$= 0 + 1(x)$$
$$\boxed{Y = x}$$

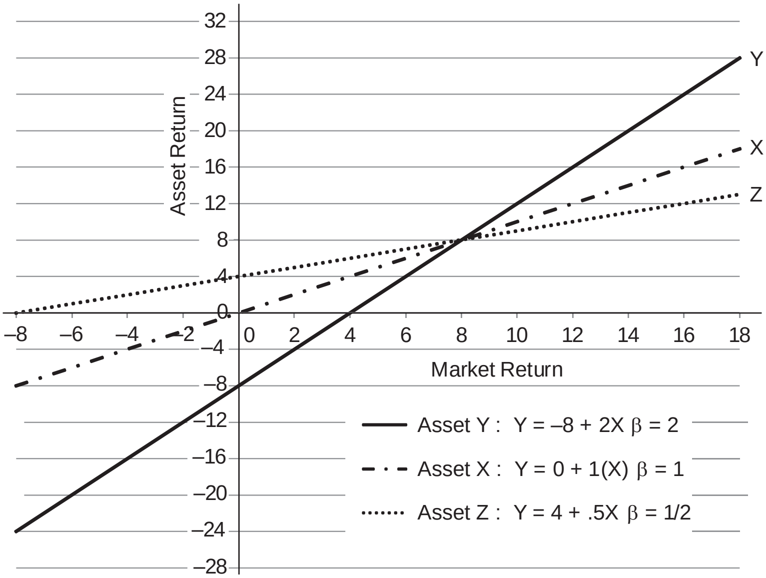

FIGURE 2.5

Asset Y would have a line with a slope of 2, an x-intercept of 4, and a y-intercept of -8:

$$=$$

$$Slopey = \frac{16-8}{12-8} = \frac{8}{4} = 2$$

The line equation would be

$$\boxed{Y = 8 + 2x}$$

Asset Z would have a line with a slope of. 5, a y-intercept of 4, and an x-intercept of -8:

$$Slopez = \frac{10-8}{12-8} = \frac{2}{4} = \frac{1}{2}$$

The line equation would be

$$\boxed{Y = 4 + .5x}$$

Since all three of these equations are linear equations of regression lines, beta can be read off the equation directly as it corresponds to the b value in the standard format of the linear equation:

$$\boxed{Y = a + bx}$$

Unit IV

Mathematics of Insurance

1. **Life Annuities**
2. **Life Insurance**
3. **Property and Casualty Insurance**

   Unit VIII Summary

   List of Formulas

   Exercises for Unit IV

4.1 Life Annuities

Life annuities are different from the annuities certain that we discussed earlier. In a major distinction, life annuities are more related to life insurance for being contingent. A **contingent contract** involves a sequence of payments that are dependent on an occurrence of a certain event that cannot be foretold. In this case, it is either death or living up to a certain age. This element of uncertainty requires the use of probability distribution, which in this context is in the form of a mortality table. Like life insurance, life annuities are dealt with by insurance companies and the person to whom the annuity would be payable, is called an annuitant, as opposed to the “insured”. Life insurance proceeds are payable to survivors upon the death of the insured.

The typical stated premium of life annuities and life insurance is usually a gross premium, which includes the loading costs in addition to the net premium. **Loading costs** include a company’s operating expenses and profit margin. The net premium is the pure cost of the ultimate benefits to the annuitant or the insured, usually broken down into installments unless it is paid in a single payment on the day of purchase. In this case it is called a **net single premium**, while a yearly installment is called a **net annual premium**. Our calculations here focus on the net premium but without the loading component, since the loading would vary from one company to another. This is why we assume that the present value of the net premium would equal the present value of all future benefits.

4.1.1 MORTALITY TABLE

A **mortality table** contains statistical data on people living and dying categorized primarily by age and sometimes by gender. The table’s major use is in calculating predictions as to how long a person will live and when he or she will probably die, to be used in estimating the benefits for life annuities and life insurance. The first mortality table in the United States, called the *American Experience Table of Mortality* , was published in New York in 1868. The table we use in this book, the commutation table (Table 10 in the Appendix), is a sample included for the purpose of calculation. It is constructed based on U.S. Internal Revenue Service (IRS) data. We include age group 0 to 100, although the IRS data goes to age 110. Following are some major definitions of the table items:

x: the age of a person in years. Age zero is the base sample of 100,000 infants under 1 year of age.

l_x: the number of people alive at age x.

d_x: the number of people who die between age x and age $x+1$. It can be calculated as the difference between l_x and l_{x+1}.

$$\boxed{d_x = I_x - I_{x+1}}$$

Example 1.1.1 The number of people who would die at age 20 (d_{20}) is 118. It is calculated as the difference between people who are alive at age 20 and those who made it to age 21:

$$\begin{aligned} d_x &= l_x - l_{x+1} \\ d_{20} &= l_{20} - l_{21} \\ &= 97{,}741 - 97{,}623 = 118 \end{aligned}$$

l_{x+1} : the number of people alive in the age group a year after any age x. It can be calculated as the difference between the number of people living at age $x(l_x)$ and the number of people dying at age $x(d_x)$.

$$\boxed{I_{x+1} = I_x - d_x}$$

Example 1.1.2 l_{x+1} for age 15 is $l_{15\ 1+} = l_{16}$. It is 98,129, and can be calculated as

$$\begin{aligned} l_{x+1} &= 1_x - d_x \\ l_{16} &= 1_{15} - d_{15} \\ &= 98{,}196 - 67 = 98{,}129 \end{aligned}$$

q_x : the probability that a person will die between age x and $x+1$. It is called the **mortality rate** and calculated by dividing the number of people who will die between the ages of x and $x+1(d_x)$ by the number of living people in that age group (l_x):

$$\boxed{q_x = \frac{d_x}{I_x}}$$

Knowing this probability in advance would enable us to obtain either d_x or l_x in terms of the other.

$$\boxed{d_x = q_x I_x}$$

and

$$\boxed{I_x = \frac{d_x}{q_x}}$$

Example 1.1.3 q_x for age 30 is. 001316. It is calculated as

$$\begin{aligned} q_{30} &= \frac{d_{30}}{l_{30}} \\ &= \frac{127}{96{,}477} \\ &= .001316 \end{aligned}$$

In some tables, this result would be written as multiplied by 1,000 and the category would be called $(1{,}000q_x)$, which would be 1.316 $(.001316 \times 1{,}000)$. Consequently, we can apply the second formulas for calculating d_x and l_x:

$$\begin{aligned} d_{30} &= q_{30} \cdot l_{30} \\ &= .001316 \times 96{,}477 \\ &= 127 \end{aligned}$$

and

$$\begin{aligned} l_{30} &= \frac{d_{30}}{q_{30}} \\ &= \frac{127}{.001316} \\ &= 96{,}477 \end{aligned}$$

Since q_x was the probability of dying, we can also calculate the probability of living one more year at any point x. That is how probable a person age x is to live to age $x+1$. It is denoted by p_x and calculated as

$$\boxed{p_x = \frac{I_{x+1}}{I_x}}$$

Example 1.1.4 The probability that a 30-year-old person will live one more year is nearly 100%:

$$p_{30} = \frac{l_{31}}{l_{30}}$$

$$= \frac{96{,}350}{96{,}477}$$

$$= .998684$$

It is the complement of q_{30}, which was. 001316. The complementarity relation between p_x and q_x simply means that there would be one of two possibilities for a person either to live or to die:

$$\boxed{q_x + p_x = 1}$$

$$q_x + p_{30} = .001316 + .998684 = 1$$

This would give us another formula for q_x and p_x:

$$p_x = 1 - q_x$$

and

$$q_x = 1 - p_x$$

The last formula would be a proof of the previous q_x formula:

$$q_x = 1 - p_x$$

$$q_x = 1 - \frac{lx+1}{Lx}$$

$$q_x = \frac{lx - lx+1}{lx}$$

Since $l_{x+1} = l_x - d_x$, then

$$q_x = \frac{lx - lx+dx}{lx}$$

$$q_x = \frac{dx}{lx}$$

Now, we can turn to p_x, the probability of a person age x to live to age $x + 1$:

$$p_x = \frac{lx+1}{lx}$$

We can, by the same logic, say that if n is a number of years, the probability of a person age x to live to that n number of years would be

$$\boxed{np_x = \frac{I_{x+n}}{I_x}}$$

Example 1.1.5 If we want to find how probable it is that a person age 50 will live to age 70, we are talking about n as 20 years, from age 50 to age 70, and therefore the probability would be calculated as

$$\begin{aligned} {}_{20}p_{50} &= \frac{l_{50+20}}{l_{50}} = \frac{l_{70}}{l_{50}} \\ &= \frac{68{,}248}{91{,}526} \\ &= 74.6\% \end{aligned}$$

Because of the complementarity between p_x and q_x, we can conclude that

$$\begin{aligned} {}_np_x + {}_nq_x &= 1 \\ {}_nq_x &= 1 - {}_np_x \\ {}_nq_x &= 1 - \frac{l_{x+n}}{l_x} \end{aligned}$$

$$\boxed{{}_nq_x = \frac{l_x - l_{x+n}}{l_x}}$$

which is defined as the probability that a person age x will die within n years or at age $x + n$.

Example 1.1.6 What is the probability that a person age 50 will die within the next 20 years?

Here also, n is 20, and therefore the probability of death would be calculated as

$$\begin{aligned} {}_nq_x &= \frac{l_x - l_{x+n}}{l_x} \\ {}_{20}q_{50} &= \frac{l_{50} - l_{50+20}}{l_{50}} \\ &= \frac{l_{50} - l_{70}}{l_{50}} \\ &= \frac{91{,}526 - 68{,}248}{91{,}526} \\ &= 25.4\% \end{aligned}$$

We can confirm that by

$$\begin{aligned} {}_np_x + {}_nq_x &= 1 \\ {}_{20}p_{50} + {}_{20}q_{50} &= 1 \\ .746 + .254 &= 1 \end{aligned}$$

4.1.2 COMMUTATION TERMS

The mortality table also includes other terms that are used in the calculation of life annuities and life insurance: D_x, N_x, C_x, and M_x.

D_x : the present value of $1.00 for each person alive at age x. Collectively for each age group it would be calculated by multiplying the present value for $1.00 ($v^x$) or $1/(1+r)^x$ by the number of living people in age group x:

$$\boxed{D_x = l_x \cdot v^x}$$

$$\boxed{D_x = \frac{l_x}{(1+r)^x}}$$

Note that our mortality table is based on a 5% rate of interest.

Example 1.2.1 The D_x value for a person at age 40 in Table 10 is $13,483.83. It is obtained by

$$\begin{aligned} D_x &= l_x \cdot v^x \\ D_{40} &= l_{40} \cdot v^{40} \\ &= 94{,}926 \left[\frac{1}{(1+.05)^{40}} \right] \\ &= 13{,}483.83 \end{aligned}$$

We can also obtain the v^x value from the v^n table, which happens to be .14204568.

$$\begin{aligned} D_{40} &= 94{,}926(.14204568) \\ &= 13{,}483.83 \end{aligned}$$

N_x : the present value of annuity of payments for all persons living at each age group from x to the end of the table, age 100. So it can be viewed as the summation of D's above.

$$N_x = D_x + D_{x+1} + D_{x+2} + \cdot \cdot \cdot + D_{x+100}$$

$$\boxed{N_x = \sum_{k=x}^{x+100} D_k}$$

Example 1.2.2 Suppose that the annuity for each person in the cohort of 70 years is $500 per year. Then the present value of those annuities back at age zero would be calculated by multiplying the annuity by the N_{70} value of the table:

$$500N_{70} = 500(21{,}427.28)$$
$$= 10{,}713{,}640$$

C_x : the present value for \$1.00 of a payment to the beneficiaries of people who die at age x. It is calculated by multiplying the number of people who will die at age x, (d_x) by the present value of \$1.00 for the time x+1.

$$\boxed{C_x = d_x \cdot v^{x+1}}$$

Example 1.2.3 The C_x value for age 55 is 51.86 in Table 10. It is calculated as

$$C_{55} = d_{55}v^{55+1}$$
$$= d^{55}v^{56}$$
$$= 797\left[\frac{1}{(1+.05)56}\right]$$
$$= 51.86$$

Using the table value of v^{56} from the v^n table would give the same result:

$$C_{55} = 797(9.06507276)$$
$$= 51.86$$

M_x : the summation of all C_x. It represents the present value of a \$1.00 payment for all people who eventually die but are still alive at age x. It is like N_x, an accumulation of the values back from age x to age 0.

$$M_x = C_x + C_{x+1} + C_{x+3} + \cdot\ \cdot\ \cdot + C_{x+100}$$

$$\boxed{M_x = \sum_{k=x}^{x+100} C_k}$$

Example 1.2.4 If the payment for each person who dies in the 85-year age group is \$1,000, the accumulated present value for all back at age 0 would be

$$1{,}000(329.76) = 329{,}760$$

where 329.76 is M_{85} in Table 10.

L.E.: the life expectancy at each age group.

4.1.3 PURE ENDOWMENT

A **pure endowment** is a single payment received at a certain future time by a person who has to be alive at that time. Since this payment is received at a future time, we should be concerned about its present value. That is why it is equivalent to the discounted value of the payment. If we assume that nE_x is the contribution of $1.00 for n years by each person age x, the total premiums would be multiplying that contribution by the number of people alive at age x $(l_x \, .nE_x)$, and if we discount it to age x since it will be received at age $x + n$ (see Figure 1.1), we get

$$l_x \cdot nE_x = \frac{\square\square+\square}{(1+\square)^{\square}}$$

$$nE_x = \frac{\left[\frac{\square\square+\square}{(1+\square)^{\square}}\right]}{\square\square}$$

$$nE_x = \frac{\square\square+\square\left[\frac{l}{(1+1)^{\square}}\right]}{\square\square}$$

$$nE_x = \frac{\square\square+\square \cdot \square^{\square}}{\square\square}$$

This is for a hypothetical $1.00. If it is for any other actual amount of p, the present value would be

$$\boxed{nEx = P\left(\frac{I_{x+n}}{I_x}\right) v_n}$$

and for more simplicity, we can use the commutation values:

$$nE_x = \frac{l_{x+n} \cdot v^n \cdot v^x}{l_x \cdot v^x} = \frac{l_{x+n} \cdot v^{x+n}}{l_x \cdot v^x}$$

and that would be

$$nE_x = \frac{D_{x+n}}{D_x}$$

and for payment P

$$\boxed{nEx = P\left(\frac{D_{x+n}}{D_x}\right)}$$

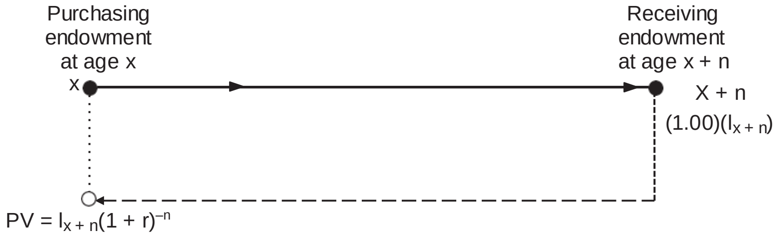

FIGURE 1.1

Example 1.3.1 Glenn is 55 years old. He would like to receive a pure endowment of $50,000 when he retires at 65. How much would he have to pay now if the rate is 5%?

The time between purchasing and receiving the endowment is $n = 10$.

$$nE_x = P\,\frac{l_{x+n}}{l_x}\,v^n$$

$$10E_{55} = 50{,}000\left(\frac{l_{55+10}}{l_{55}}\right)v^{10}$$

$$= 50{,}000\left(\frac{77{,}107}{88{,}348}\right)(0.61391325)$$

$$= 26{,}790$$

and we can also use the commutation formula:

$$nE_x = P\left(\frac{D_{x+n}}{D_x}\right)$$

$$10E_{55} = 50{,}000\left(\frac{D_{65}}{D_{55}}\right)$$

$$= 50{,}000\left(\frac{3{,}234.37}{6{,}036.50}\right)$$

$$= 26{,}790$$

4.1.4 TYPES OF LIFE ANNUITIES

The life annuities discussed here are called **single-life annuities**, referring to their distinct characteristic: that they are for a single living person and they cease when the person dies. There are two major types, whole life annuities and temporary life annuities.

Whole Life Annuities

Whole life annuities are paid to the annuitant as long as he or she lives and do not stop until the person dies. These annuities come in three categories: ordinary, due, and deferred.

Ordinary Whole Life Annuity This type of annuity, also called an **immediate whole life annuity** is distinguished by the receipt of its payments at the end of each year of the annuitant's life. We can look at these payments as a series of individual pure endowments, and for this reason, let's consider a as the present value of \$1.00 a year for each person in the age group x. The collective endowments would be

$$a_x = 1E_x + 2E_x + 3E_x + \cdot \cdot \cdot + 100E$$

$$a_x = \frac{D_{x+1}}{D_x} + \frac{D_{x+2}}{D_x} + \frac{D_{x+3}}{D_x} + \cdots + \frac{D_{x+100}}{D_x}$$

$$a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \cdot \cdot \cdot + D_{x+100}}{D_x}$$

$$a_x = \frac{N_{x+1}}{D_x}$$

and if the payment is P instead of \$1.00, then

$$\boxed{a_x = P\left(\frac{N_{x+1}}{D_x}\right)}$$

Example 1.4.1 Nicole is 45 years old. She is thinking of buying a whole life annuity that would pay her \$2,000 at the end of each year for the rest of her life. What would the premium be for this annuity?

We immediately knew that this is an ordinary whole life annuity since the payments are to be made at the end of each year.

$$a_x = P\left(\frac{N_{x+1}}{D_x}\right)$$

$$a_{45} = 2{,}000\left(\frac{N_{46}}{D_{45}}\right)$$

$$= 2{,}000\left(\frac{154{,}684.29}{10{,}417.24}\right)$$

$$= 29{,}697.75$$

Whole Life Annuity Due This is the same as a whole life annuity except that the first payment is made at the time of purchase and continues to be made after that on the same time of each year for the rest of the annuitant's life. Since we designated our formulas based on a $1.00 payment, and we stated that this annuity has the first payment made one year earlier, we can conclude that the whole life annuity due $(\ddot{a}_x)$ is different from the ordinary whole life annuity (a_x):

$$\ddot{a}_x = 1 + a_x$$

$$\ddot{a}_x = 1 + \frac{N_{x+1}}{D_x}$$

$$\ddot{a}_x = \frac{D_x + N_{x+1}}{D_x}$$

and since $N_{x+1} = D_{x+1} + D_{x+2} + \cdot\ \cdot\ \cdot + D_{x+100}$, then

$$\ddot{a}_x = \frac{D_x + D_{x+1} + D_{x+2} + \cdot\ \cdot\ \cdot + D_{x+100}}{D_x}$$

which made the numerator N_x:

$$\ddot{a}_x = \frac{N_x}{D_x}$$

and for payment P instead of $1.00, we obtain the formula for the whole life annuity due:

$$\boxed{\ddot{a}_x = P\left(\frac{N_x}{D_x}\right)}$$

Example 1.4.2 Suppose that Nicole of Example 1.4.1 wanted to get her first payment right at the time of purchase. Would that affect the amount that she would receive each year?

This change would make the annuity an annuity due, and therefore

$$\ddot{a}_{45} = P\left(\frac{N_x}{D_x}\right)$$

$$= 2{,}000\left(\frac{165{,}101.53}{10{,}417.24}\right)$$

$$= 31{,}697.75$$

So the payment got bigger! But by how much? It got bigger by

$$31{,}697.75 - 29{,}697.75 = 2{,}000$$

and that is exactly the one payment difference between the ordinary and the due.

Example 1.4.3 A 62-year-old woman who was hit by a car received an insurance settlement which she wanted to put in a whole life annuity so that an annual payment of \$12,000 is made to her starting immediately and continuing to be paid at the same time of each year for the rest of her life. How much was her settlement?

$$\ddot{a}_x = P\left(\frac{N_x}{D_x}\right)$$

$$\ddot{a}_{62} = P\left(\frac{N_{62}}{D_{62}}\right)$$

$$= 12{,}000\left(\frac{46{,}632.42}{3{,}950.12}\right)$$

$$= 12{,}000(11.81)$$

$$= 141{,}720$$

Deferred Whole Life Annuity This is an annuity whose first payment is deferred to a period (n) that is beyond one year after the purchase date. Since the purchase occurs at age x, the deferment period would be $x + n$ for the annuity due, which starts on the day of purchase. It could also be $x + 1 + n$ for the ordinary annuity, which starts one year after the purchase date, at $x + 1$ (see Figure 1.2).

The present value of the deferred ordinary whole life annuity would be $_{n|}a_x$, which can be obtained by

$$_{n|}a_x = ({}_{n+1}E_x) + ({}_{n+2}E_x) + ({}_{n+3}E_x) + \cdots + ({}_{n+100}E_x)$$

$$_{n|}a_x = \frac{D_{x+1+n} + D_{x+2+n} + D_{x+3+n} + \cdots + D_{x+100+n}}{D_x}$$

$$_{n|}a_x = \frac{N_{x+1+n}}{D_x}$$

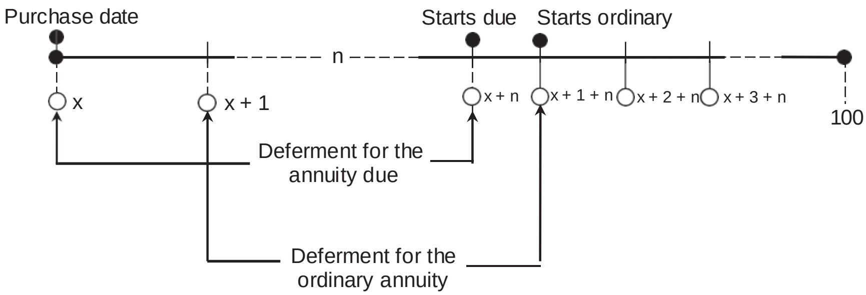

FIGURE 1.2

and for payment P:

$$\boxed{{}_{n|}a_x = P\left(\frac{N_{x+1+n}}{D_x}\right)}$$

In the same manner, we can obtain the deferred whole life annuity due $({}_{n|}\ddot{a}_x)$ by

$${}_{n|}\ddot{a}_x = {}_nE_x + ({}_{n+1}E_x) + ({}_{n+2}E_x) + \cdots + ({}_{n+100}E_x)$$

$${}_{n|}\ddot{a}_x = \frac{D_{x+n} + D_{x+1+n} + D_{x+2+n} + \cdots + D_{x+100+n}}{D_x}$$

$${}_{n|}\ddot{a}_x = \frac{N_{x+n}}{D_x}$$

and for payment P it would be

$$\boxed{{}_{n|}\ddot{a}_x = P\left(\frac{N_{x+n}}{D_x}\right)}$$

Example 1.4.4 Linda, who is 25, wishes to purchase an ordinary whole life annuity that pays \$2,400 a year, but she wants to start collecting her payments 25 years later. How much would she have to pay for the annuity?

$${}_{n|}a_x = P\left(\frac{N_{x+1+n}}{D_x}\right)$$

$${}_{25|}a_{25} = P\left(\frac{N_{25+1+25}}{D_{25}}\right)$$

$$= 2{,}400\left(\frac{N_{51}}{D_{25}}\right)$$

$$= 2{,}400\left(\frac{110{,}125.43}{28{,}676.85}\right)$$

$$= 9{,}216.53$$

Example 1.4.5 A 27-year-old man purchased a whole life annuity due so he will be able to receive \$10,000 a year when he is 65. How much would his net single premium be?

$${}_{n|}\ddot{a}_x = P\left(\frac{N_{x+n}}{D_x}\right)$$

$${}_{38|}\ddot{a}_{27} = 10{,}000\left(\frac{N_{65}}{D_{27}}\right)$$

$$38|\ddot{a}_{27} = 10{,}000\left(\frac{35{,}523.27}{25{,}942.72}\right)$$

$$= 13{,}692$$

Example 1.4.6 An inheritance of $65,000 went to a 12-year-old girl. Her guardians decided to buy her a whole life annuity that starts paying her annually when she turns 18. How much would she receive annually?

The net single premium is $65,000 and $n = 6$.

$$65{,}000 = P\left(\frac{N18}{D12}\right)$$

$$= P\left(\frac{35{,}523.27}{25{,}942.72}\right)$$

$$= P(14.30)$$

$$P = \frac{65{,}000}{14.30} = 4{,}547$$

Temporary Life Annuities

Whereas whole life annuities pay the annuitant for the rest of his or her life, temporary annuities pay only for a certain contracted period of time, given that the annuitant is alive throughout that period. When the person dies, the payments cease. Depending on the date of the first payment, temporary life annuities are also classified into ordinary, due, and deferred.

Ordinary Temporary Life Annuity If the term of annuity is n, the present value of the temporary life annuity ($a_{x:\overline{n}|}$) can be the difference between an ordinary whole life annuity (a_x) and a deferred whole life annuity ($n|a_x$):

$$a_{x:\overline{n}|} = a_x - n|a_x$$

$$a_{x:\overline{n}|} = \frac{N_{x+1}}{D_x} - \frac{N_{x+1+n}}{D_x}$$

$$a_{x:\overline{n}|} = \frac{N_{x+1} - N_{x+1+n}}{D_x}$$

and for payment P it would be

$$\boxed{a_{x:\overline{n}|} = P\left(\frac{N_{x+1} - N_{x+1+n}}{D_x}\right)}$$

Temporary Life Annuity Due In an ordinary temporary life annuity, the payments are made at the end of each year, but if the payments are made at the beginning of each year, the annuity would be considered a temporary annuity due ($\ddot{a}_{x:\overline{n}|}$), and it would be calculated as the difference between a whole life annuity due ($\ddot{a}_x$), and a deferred annuity due ($n|\ddot{a}_x$):

$$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - n|\ddot{a}_x$$

$$\ddot{a}_{x:\overline{n}|} = \frac{N_x}{D_x} - \frac{N_{x+n}}{D_x}$$

$$\ddot{a}_{x:\overline{n}|} = \frac{N_x - N_{x+n}}{D_x}$$

and for payment P it would be

$$\boxed{\ddot{a}_{x:\overline{n}|} = P\left(\frac{N_x - N_{x+n}}{D_x}\right)}$$

Example 1.4.7 A person aged 33 wants to purchase an ordinary life annuity that would pay him \$3,600 a year for 10 years. How much would his net single premium be?

Since this is specified for 10 years, it is an ordinary temporary annuity.

$$a_{x:\overline{n}|} = P\left(\frac{Nx+1 - Nx+1+n}{Dx}\right)$$

$$a_{33:\overline{10}|} = 3{,}600\left(\frac{N33+1 - N33=1=10}{D33}\right)$$

$$= 3{,}600\left(\frac{N34 - N44}{D33}\right)$$

$$= 3{,}600\left(\frac{323{,}068.92 - 176{,}076.33}{19{,}205.35}\right)$$

$$= 27{,}553.43$$

Example 1.4.8 Find the net present value of the annuity in Example 1.4.7 if the annuitant requests that the payments be made at the beginning of the year.

This request makes the annuity a temporary annuity due:

$$\ddot{a}_{x:\overline{n}|} = P\left(\frac{Nx - Nx+n}{Dx}\right)$$

$$\ddot{a}_{33:\overline{10}|} = 3{,}600\left(\frac{N33 - N43}{D33}\right)$$

$$= 3{,}600 \left(\frac{342{,}274.28 - 187{,}635.20}{19{,}205.35} \right)$$

$$= 28{,}986.75$$

Forborne Temporary Life Annuity Due Sometimes the annuitant of the temporary life annuity due chooses not to cash the payments, allowing them to accumulate for a certain period of time (n) and become a pure endowment. In this case, when the annuitant forbears to draw the annuity, it would be called a **forborne temporary life annuity** *(nF_x)*, which is calculated as

$$nF_x = \frac{N_x - N_{x+n}}{D_{x+n}}$$

and for payment P it would be

$$\boxed{nF_x = P \left(\frac{N_x - N_{x+n}}{D_{x+n}} \right)}$$

Example 1.4.9 At the start of the year, David turns 50 and is to receive $3,000 as his temporary life annuity payment, which continues until he is 60 years old. David decides to leave the money with the insurance company to accumulate for 10 years. How much will he get when he turns 60?

$$nF_x = P \left(\frac{N_x - N_{x+n}}{D_{x+n}} \right)$$

$$10F_{50} = 3{,}000 \left(\frac{N_{50} - N_{60}}{D_{60}} \right)$$

$$= 3{,}000 \left(\frac{118{,}106.84 - 55{,}329.23}{4{,}482.32} \right)$$

$$= 42{,}016.82$$

Deferred Temporary Life Annuity This is the last type of temporary life annuity, along with the ordinary and due types. Payments of this annuity will not start until a period of time (k) has elapsed from the day of annuity purchase; k is more than one year. The deferred temporary life annuity is most likely to be an annuity due, and its present value ($k|\ddot{a}_{x:\overline{n}|}$) would be calculated as

$$k|\ddot{a}_{x:\overline{n}|} = \frac{N_{x+k} - N_{x+k+n}}{D_x}$$

where k is the deferment period, x is the annuitant age, and n is the annuity time. For a payment P, the net single premium or the present value of the deferred temporary life annuity would be

$$\boxed{{}_{k|}\ddot{a}_{x:\overline{n}|} = P\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right)}$$

Example 1.4.10 Find the net single premium for a temporary life annuity that runs for 12 years paying an annual payment of $2,500 to a 15-year-old boy. This annuity comes with a stipulation that the annuitant will not start to receive the annual payments until he turns 18.

So, x is 15, k is 3 (18 − 15), and n is 12.

$$
\begin{aligned}
{}_{k|}\ddot{a}_{x:\overline{n}|} &= P\left(\frac{N_{x+k} - N_{x+k+n}}{D_x}\right) \\
{}_{3|}\ddot{a}_{15:\overline{12}|} &= 2{,}500\left(\frac{N_{18} - N_{30}}{D_{15}}\right) \\
&= 2{,}500\left(\frac{782{,}546.43 - 406{,}021.84}{47{,}233.95}\right) \\
&= 19{,}928.70
\end{aligned}
$$

4.2 Life Insurance

The fundamental difference between a life annuity and life insurance is that a life annuity pays to an annuitant whom the policy stipulates to be alive, whereas life insurance pays for the survivors of the insured upon her death. So the life insurance policy stipulates the death of the insured before paying any benefits. The insured would pay either a single premium or annual premiums in purchasing the life insurance policy. The insured or the policyholder's age would be determined by her age at the time of purchase: specifically, her age on the nearest birthday to the formal policy date from which the next years start to count. As we did with life annuities, our calculations skip the operating costs or loadings that would normally be added. We consider only the value of the net premium, which would be equal to the present value of the face of the policy. Therefore, in the case of breaking down the net single premium into annual installments, the present value of all the premiums would be equal to the net single premium.

There are three major types of life insurance policies: the whole life policy, the term policy, and the endowment policy.

4.2.1 WHOLE LIFE INSURANCE POLICY

According to a whole life insurance policy, the insurance company is obligated to pay the face value of the policy to the survivors of the policyholder upon his death, whenever it occurs. The benefits are usually paid at the end of the year in which the insured's death occurs. As we did with the life annuities, we construct our formulas based on a $1.00 present value so that we can multiply it by the face value in question. The net single premium (A_x) for this policy is the sum of the mathematical expectations that the face value would be paid to the policy beneficiaries. The mathematical expectation is the product of the probability that the insured would die (q_x), and the present value of the policy benefits (v^n):

$$A_x = q_x v^1 + q_{x+1} v^2 + q_{x+2} v^3 + \cdot \cdot \cdot$$

$$A_x = (\frac{d_x}{l_x})v^1 + (\frac{d_{x+1}}{l_x})v^2 + (\frac{d_{x+2}}{l_x})v^3 + \cdot \cdot \cdot$$

If we multiply the numerator and denominator by v^x, we get

$$A_x = \frac{d_x v^{x+1}}{1_x v^x} + \frac{d_{x+1} v^{x+2}}{1_x v^x} + \frac{d_{x+2} v^{x+3}}{1_x v^x} + \cdots$$

$$A_x = \frac{d_x v^{x+1} + d_{x+1} v^{x+2} + d_{x+2} v^{x+3} + \cdots}{l_x v^x}$$

Since $d_x v^{x+1} = C_x$ and $l_x v^x = D_x$, then

$$A_x = \frac{C_x + C_{x+1} + C_{x+2} + \cdots}{D_x}$$

and since $M_x = \sum C_k$, then

$$A_x = \frac{M_x}{D_x}$$

and for a certain face value of the policy (F), the net single premium or, as it is sometimes called, the **cost of insurance** would be

$$\boxed{A_x = F\left(\frac{M_x}{D_x}\right)}$$

Example 2.1.1 What would be the net single premium for a whole life insurance policy for Tim, who is 49 years old and wants his family to receive $200,000 after his death?

$$A_x = F\left(\frac{M_x}{D_x}\right)$$

$$A_{49} = 200{,}000 \left(\frac{M_{49}}{D_{49}}\right)$$

$$= 200{,}000 \left(\frac{2{,}400.44}{8{,}425.80}\right)$$

$$= 56{,}978.33$$

4.2.2 ANNUAL PREMIUM: WHOLE LIFE BASIS

Paying $56,978.33 at once as in the example above would be difficult for most people to afford. That is why companies break this single premium into annual premiums that would be paid by the insured until his death. Having a large number of payments in this case would lead to having a whole life annuity due

($\ddot{a}_x$). To calculate the annual premium for a whole life insurance policy (P_x), we set up the following:

$$A_x = P_x \ddot{a}_x$$
$$P_x = \frac{A_x}{\ddot{a}_x} = \frac{M_x/D_x}{N_x/D_x}$$
$$P_x = \frac{M_x}{D_x} \cdot \frac{D_x}{N_x}$$
$$P_x = \frac{M_x}{N_x}$$

and for face value F:

$$\boxed{P_x = F\left(\frac{M_x}{N_x}\right)}$$

Example 2.2.1 Find the annual premium for Tim in Example 2.1.1 since he cannot afford to pay the entire net premium in one payment.

$$P_x = F\left(\frac{M_x}{N_x}\right)$$
$$P_{49} = 200{,}000\left(\frac{M_{49}}{N_{49}}\right) = \frac{2{,}400.44}{126{,}532.64}$$
$$= 200{,}000\left(\frac{2{,}400.44}{126{,}532.64}\right)$$
$$= 3{,}794.18$$ annual premium for a $200,000 whole life policy

4.2.3 ANNUAL PREMIUM: *m*-PAYMENT BASIS

If the insured wants to limit the number of premiums to a certain number instead of continuing to pay until his death, this can also be arranged with the insurance company. If we consider m to be the number of years to which the net premium will be paid, it will form a temporary life annuity due ($\ddot{a}_{x:\overline{m}|}$), and the annual premium (mP_x) would be obtained by

$$A_x = mP_x \ddot{a}_{x:\overline{m}|}$$
$$mP_x = \frac{A_x}{\ddot{a}_{x:\overline{m}|}}$$

$$mP_x = \frac{M_x/D_x}{(N_x - N_{x+m})/D_x}$$

$$mP_x = \frac{M_x}{D_x} \cdot \frac{D_x}{N_x - N_{x+m}}$$

cancelling D_x out:

$$mP_x = \frac{M_x}{N_x - N_{x+m}}$$

and for face value F it would be

$$\boxed{mP_x = F\left(\frac{M_x}{N_x - N_{x+m}}\right)}$$

Example 2.3.1 Suppose that Tim of earlier examples wants to pay all his net premium in 10 years. How much would he be paying annually?

$$\begin{aligned} mP_x &= F\left(\frac{M_x}{N_x - N_{x+m}}\right) \\ 10P_{49} &= 200{,}000\left(\frac{M_{49}}{N_{49} - N_{49+10}}\right) \\ &= 200{,}000\left(\frac{2{,}400.44}{126{,}532.64 - 60{,}095.41}\right) \\ &= 7{,}226.19 \end{aligned}$$

This is the annual premium if the entire cost of insurance as it is paid in only 10 years. It would be called a 10-payment premium.

4.2.4 DEFERRED WHOLE LIFE POLICY

A deferred whole life insurance policy is based on the notion that the insured will not die until after a certain period of time (n), called the **deferment period**. The net single premium of this policy for \$1.00 is denoted by $n|A_x$ and determined by

$$n|A_{x\,1} = \frac{d_{x+n} \cdot v^{n+1}}{l_x} + \frac{d_{x+n+1} \cdot v^{n+2}}{l_x} + \frac{d_{x+n+2} \cdot v^{n+3}}{l_x} + \cdots$$

$$n|A_{x\,1} = \frac{d_{x+n} \cdot v^{n+1} + d_{x+n+1} \cdot v^{n+2} + d_{x+n+2} \cdot v^{n+3} + \cdots}{l_x}$$

Multiplying by v^x, we get

$$n|A_x = \frac{d_{x+n} \cdot v^{x+n+1} + d_{x+n+1} \cdot v^{x+n+2} + d_{x+n+2} \cdot v^{x+n+3} + \cdots}{l_x \cdot v^x}$$

$$n|A_x = \frac{C_{x+n} + C_{x+n+1} + C_{x+n+2} + \cdots}{D_x}$$

$$n|A_x = \frac{M_{x+n}}{D_x}$$

and for face value F it would be

$$\boxed{n|A_x = F\left(\frac{M_{x+n}}{D_x}\right)}$$

Example 2.4.1 Rita is 35. She wants to buy a $30,000 life insurance policy that would be activated only if she dies when she is 40 or older. How much would her net single premium be?

The period of deferment (n) is 5 years.

$$n|A_x = F\left(\frac{M_{x+n}}{D_x}\right)$$

$$5|A_{35} = 30{,}000\left(\frac{M_{35+5}}{D_{35}}\right)$$

$$= 30{,}000\left(\frac{2{,}717.07}{17{,}369.06}\right)$$

$$= 4{,}692.95$$

4.2.5 DEFERRED ANNUAL PREMIUM: WHOLE LIFE BASIS

We can also calculate the cost of this insurance policy in terms of its annual premiums $(n|P_x)$ involving the annuity due formed by these premiums:

$$n|A_x = n|P_x \cdot \ddot{a}_x$$

$$n|P_x = \frac{n|A_x}{\ddot{a}_x}$$

$$n|P_x = \frac{M_{x+n}/D_x}{N_x/D_x}$$

$$n|P_x = \frac{M_{x+n}}{D_x}\left(\frac{D_x}{N_x}\right)$$

cancelling D_x out:

$$n|P_x = \frac{M_{x+n}}{N_x}$$

and for face value F it would be

$$\boxed{n|P_x = F\left(\frac{M_{x+n}}{N_x}\right)}$$

Example 2.5.1 Suppose that Rita wants to break the premium into annual payments. How much would each payment be?

$$n|P_x = F\left(\frac{M_{x+n}}{N_x}\right)$$

$$5|P_{35} = 30{,}000\left(\frac{M_{35+5}}{N_{35}}\right)$$

$$= 30{,}000\left(\frac{2{,}717.07}{304{,}804.19}\right)$$

$$= 267.42$$

4.2.6 DEFERRED ANNUAL PREMIUM: *m*-PAYMENT BASIS

The net single premium of this policy can also be broken down into a certain number of annual payments corresponding to m-period during which the policyholder must live. Let's suppose that the annual premium based on this m years is $mP(n|A_x)$, which can be calculated as

$$n|A_x = mP(n|A_x) \cdot \ddot{a}_{x:\overline{m}|}$$

$$mP(n|A_x) = \frac{n|A_x}{\ddot{a}_{x:\overline{m}|}}$$

$$mP(n|A_x) = \frac{M_{x+n}/D_x}{(N_x - N_{x+m})/D_x}$$

$$mP(n|A_x) = \frac{M_{x+n}}{D_x}\,\frac{D_x}{N_x - N_{x+m}}$$

cancelling out D_x:

$$mP(n|A_x) = \frac{M_{x+n}}{N_x - N_{x+m}}$$

and for face value F it would be

$$\boxed{mP(n|A_x) = F\left(\frac{M_{x+n}}{N_x - N_{x+m}}\right)}$$

Example 2.6.1 Let's suppose that Rita wants to pay her net premium in 4 years. How much would the annual premium be?

$$mP(n|A_x) = F\left(\frac{M_{x+n}}{N_x - N_{x+m}}\right)$$

$$4P(5|A_{35}) = 30{,}000\left(\frac{M_{35+5}}{N_{35} - N_{35+4}}\right)$$

$$= 30{,}000\left(\frac{2{,}717.07}{304{,}804.19 - 240{,}290.14}\right)$$

$$= 1{,}263.48$$

With this amount annually, Rita would be able to pay all the policy cost in 4 years, provided that she lives through these 4 years.

4.2.7 TERM LIFE INSURANCE POLICY

A term life insurance policy would pay the face value of the policy to the survivors only when the insured dies within a specified period of time called the **term** of the policy (n). Technically, the net single premium of this policy $(A^1_{x:\overline{n}|})$ is viewed as the difference between the costs of whole life insurance (A_x) and deferred whole life insurance $(n|A_x)$.

$$A^1_{x:\overline{n}|} = A_x - n|A_x$$

$$A^1_{x:\overline{n}|} = \frac{M_x}{D_x} - \frac{M_{x+n}}{D_x}$$

$$A^1_{x:\overline{n}|} = \frac{M_x - M_{x+n}}{D_x}$$

and for face value F it would be

$$\boxed{A^1_{x:\overline{n}|} = F\left(\frac{M_x - M_{x+n}}{D_x}\right)}$$

Example 2.7.1 Alison is 57. She would like to buy a 13-year term life insurance policy of \$50,000. How much would it cost her?

$$A^1_{x:\overline{n}|} = F\left(\frac{M_x - M_{x+n}}{D_x}\right)$$

$$A^1_{57:\overline{13}|} = 50{,}000\left(\frac{M_{57} - M_{57+13}}{D_{57}}\right)$$

$$= 50{,}000\left(\frac{2{,}014.22 - 1{,}222.70}{5{,}372.84}\right)$$

$$= 7{,}365.94$$

Term insurance policies are usually paid for by annual payments where the number of payments (k) should be either equal to or less than the term of the policy (n).

$$\boxed{k \leq n}$$

The annual premium would be $kP^1_{x:\overline{n}|}$, which is calculated as

$$A^1_{x:\overline{n}|} = kP^1_{x:\overline{n}|} \cdot \ddot{a}_{x:\overline{k}|}$$

$$kP^1_{x:\overline{n}|} = \frac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{k}|}}$$

$$kP^1_{x:\overline{n}|} = \frac{(M_x - M_{x+n})/D_x}{(N_x - N_{x+k})/D_x}$$

$$kP^1_{x:\overline{n}|} = \frac{M_x - M_{x+n}}{D_x} \cdot \frac{D_x}{N_x - N_{x+k}}$$

cancelling out D_x:

$$kP^1_{x:\overline{n}|} = \frac{M_x - M_{x+n}}{N_x - N_{x+k}}$$

and for face value F it would be

$$\boxed{kP^1_{x:\overline{n}|} = F\left(\frac{M_x - M_{x+n}}{N_x - N_{x+k}}\right)}$$

Example 2.7.2 Find the annual premium for Alison's term life insurance if she wants to pay it off in 10 years.

$$kP^1_{x:\overline{n}|} = F\left(\frac{M_x - M_{x+n}}{N_x - N_{x+k}}\right)$$

$$10P^{\,1}_{57:\overline{13}|} = 50{,}000\left(\frac{M_{57} - M_{70}}{N_{57} - N_{67}}\right)$$

$$= 50,000\left(\frac{2,014.22 - 1,222.70}{70,531 - 29,271.95}\right)$$

$$= 959.21$$

Note that if the number of payments (k) is equal to the term of the policy (n), the formula can be adjusted accordingly:

$$\boxed{nP^{1}_{x:\overline{n}|} = F\left(\frac{M_x - M_{x+n}}{N_x - N_{x+n}}\right) \qquad \text{when } k = n}$$

Example 2.7.3 Find the annual premium for Alison in Example 2.7.2 if she wants to pay out her insurance cost in annual premiums until her policy term is over.

This means that she would pay off the cost in 13 payments ($n = k = 13$).

$$nP^{1}_{x:\overline{n}|} = F\left(\frac{M_x - M_{x+n}}{N_x - N_{x+n}}\right)$$

$$13P^{\,1}_{57:\overline{13}|} = 50{,}000\left(\frac{M_{57} - M_{70}}{N_{57} - N_{70}}\right)$$

$$= 50{,}000\left(\frac{2{,}014.22 - 1{,}222.70}{70{,}531 - 21{,}427.28}\right)$$

$$= 805.97$$

4.2.8 ENDOWMENT INSURANCE POLICY

The endowment insurance is, in fact, a combination of a term life insurance and a pure endowment where both have the same term (n). This mix is designed to assure the receipt of the policy benefits whether the insured dies or lives throughout the specified term. Therefore, the face value of the policy would be paid at any rate. It would be paid to the insured if she survives until the end of the term specified, and it would be paid to the survivors if she dies within the term specified. Therefore, the net single premium of the endowment insurance policy $(A_{x:\overline{n}|})$ would be the sum of the net single premium for the n-term life insurance $(A^{1}_{x:\overline{n}|})$, and the net single premium for the n-term pure endowment (nE_x):

$$A^{1}_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + nE_x$$

$$A^{1}_{x:\overline{n}|} = \frac{M_x - M_{x+n}}{D_x} + \frac{D_{x+n}}{D_x}$$

$$A^{1}_{x:\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}$$

and for face value F it would be

$$\boxed{A^{1}_{x:\overline{n}|} = F\left(\frac{M_x - M_{x+n} + D_{x+n}}{D_x}\right)}$$

Example 2.8.1 Dan is 46. He has just purchased a \$65,000 endowment policy with a 15-year term. How much would this policy cost him?

$$A^{1}_{x:\overline{n}|} = F\left(\frac{M_x - M_{x+n} + D_{x+n}}{D_x}\right)$$

$$A^{1}_{46:\overline{15}|} = 65{,}000\left(\frac{M_{46} - M_{46+15} + D_{46+15}}{D_{46}}\right)$$

$$= 65{,}000\left(\frac{2{,}518.91 - 1{,}789.21 + 4{,}210.49}{9{,}884.83}\right)$$

$$= 32{,}485.37$$

4.2.9 ANNUAL PREMIUM FOR THE ENDOWMENT POLICY

Just like with other policies, the large net single premium can be paid off in annual installments that would form an annuity due and result into adjusting the formula as follows:

1. If the number of annual premiums (k) is equal to the policy term (n), $k = n$, then

$$\boxed{P_{x:\overline{n}|} = F\left(\frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}\right)}$$

2. If the number of annual premiums (k) is less than the policy term (n), $k < n$, then:

$$\boxed{kP_{x:\overline{n}|} = F\left(\frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+k}}\right)}$$

Note that k cannot be larger than n; k is always either equal or less than n [$k \leq n$].

Example 2.9.1 Find the annual premiums for the endowment policy in Example 2.8.1 as the cost is paid off along the policy term.

$$P_{x:\overline{n}|} = F\left(\frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}\right)$$

$$P_{46:\overline{15}|} = 65{,}000\left(\frac{M_{46} - M_{61} + D_{61}}{N_{46} - N_{61}}\right)$$

$$= 65{,}000\left(\frac{2{,}518.91 - 1{,}789.21 + 4{,}210.49}{154{,}684.29 - 50{,}846.91}\right)$$

$$= 3{,}092.45$$

Example 2.9.2 Suppose that Dan wants to pay off his net single premium in 10 years. How much would his annual premium be?

Now $k = 10$, which is less than the policy term, n (15).

$$kP_{x:\overline{n}|} = F\left(\frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+k}}\right)$$

$$10P_{46:\overline{15}|} = 65{,}000\left(\frac{M_{46} - M_{61} + D_{61}}{N_{46} - N_{56}}\right)$$

$$= 65{,}000\left(\frac{2{,}518.91 - 1{,}789.21 + 4{,}210.49}{154{,}684.29 - 76{,}228.19}\right)$$

$$= 4{,}092.89$$

4.2.10 LESS THAN ANNUAL PREMIUMS

In practice, insurance companies agree to collect their net single premium not only in annual installments but also quarterly, monthly, or other terms. This increases the cost to the policyholder, but people can often afford a small premium despite the expense attached to such a convenience. The additional cost to the premium can be figured out as an added certain percentage (j). So if we denote the less than annual premium by $p(m)$ and the annual premium by P,

$$\boxed{P^{(m)} = \frac{P(1 + j)}{m}}$$

Example 2.10.1 Let's suppose that Dan, who we met earlier, cannot afford to pay his annual premium of $3,092.45 and asks the company to let him pay whatever they deem appropriate every other month, that is, in six payments a year. Given that the company charges $6\frac{1}{2}$% extra, determine how much his bimonthly premium would be.

$P = \$3{,}092.45; j = 6\frac{1}{2}\% = .065.$

$$P^{(m)} = \frac{P(1+j)}{m}$$

$$P^{(6)} = \frac{3{,}092.45(1+.065)}{6}$$

$$= 548.91$$

4.2.11 NATURAL PREMIUM VS. THE LEVEL PREMIUM

The natural premium is the net single premium for a one-year term life insurance that pledges to pay the face value of the policy only when the insured dies within one year. We can obtain such a premium by adjusting the regular term life insurance formula $(A^1_{x:\overline{n}|})$ by making $n = 1$:

$$NA^1_{x:\overline{1}|} = \frac{M_x - M_{x+1}}{D_x}$$

$$NA^1_{x:\overline{1}|} = \frac{C_x}{D_x}$$

and for face value F it would be

$$\boxed{NA^1_{x:\overline{1}|} = F\left(\frac{C_x}{D_x}\right)}$$

Example 2.11.1 What would be the natural premium for a 75-year-old person who buys a \$100,000 term life insurance policy?

$$NA^1_{75:\overline{1}|} = F\left(\frac{C_{75}}{D_{75}}\right)$$

$$= 100{,}000\left(\frac{62.78}{1{,}462.66}\right)$$

$$= 4{,}292.18$$

Since the natural premium is obtained by dividing C_x by D_x, it tends to increase with age. This is because C_x increases and D_x decreases as people advance in age. Normally, the natural premium for one year is supposed to be sufficient to cover all death claims at the end of that year, but this becomes problematic, as the natural premium rises dramatically with older age. One way to deal with this problem and make life insurance affordable is to rely on the regular annual premium, which we described as a level premium. Level premium policies have a higher premium than that of a natural premium in the early years and a lower premium in the later years. On balance, the early years would create an excess fund that would be enough, with the interest it accumulates, to face the deficit that the later years create. In Table 2.1 and Figure 2.1 we show how a natural premium interacts with a level premium to creat that excess and deficit. We take, for example, a 15-year term insurance policy for $100,000 issued to a 50-year-old. We calculate the level premium for the policy in the usual way:

$$
\begin{aligned}
{}_nP_x &= (F)\left(\frac{M_x - M_{x+n}}{N_x - N_{x+n}}\right) \\
{}_{15}P_{50} &= 100{,}000\left(\frac{M_{50} - M_{50+15}}{N_{50} - N_{50+15}}\right) \\
&= 100{,}000\left(\frac{2{,}357.27 - 1{,}542.78}{118{,}106.84 - 35{,}523.27}\right) \\
&= 986.26
\end{aligned}
$$

which would be fixed for the entire term of 15 years, from age 50 to age 64. We also calculate the natural premium for each year by multiplying the face value of the policy, $100,000, by the ratio of C_x to D_x. We see in Table 2.1 that the level premium is higher than the natural premium between ages 50 and 56, and lower between ages 57 and 64. This discrepancy creates the excess and deficit areas on Figure 2.1. The straight line of the level premium first passes above the

TABLE 2.1 Natural Premiums vs. Level Premiums in a $100,000 15-Year Term Insurance Policy at 5%

	(1) Age	(2) C_x	(3) D_x	(4) Level Premium, ${}_{15}P_{50}$	(5) Natural Premium 100,000 [(2) · (3)]	(6) Excess/ Deficit [(4) – (5)]
1	50	44.85	7,981.41	986.26	561.93	424.33
2	51	46.19	7,556.49	986.26	611.26	375.00
3	52	47.53	7,150.47	986.26	664.71	321.55
4	53	49.07	6,762.44	986.26	725.62	260.64
5	54	50.49	6,391.34	986.26	789.97	196.29
6	55	51.86	6,036.50	986.26	859.11	127.15
7	56	53.05	5,697.19	986.26	931.16	55.10
8	57	54.24	5,372.84	986.26	1,009.52	−23.26
9	58	55.48	5,062.75	986.26	1,095.85	−109.59
10	59	56.91	4,766.18	986.26	1,194.04	−207.78
11	60	58.38	4,482.32	986.26	1,302.45	−316.19
12	61	59.87	4,210.49	986.26	1,421.92	−435.66
13	62	61.23	3,950.12	986.26	1,550.08	−563.82
14	63	62.32	3,700.79	986.26	1,683.96	−697.70
15	64	63.00	3,462.24	986.26	1,819.63	−833.37

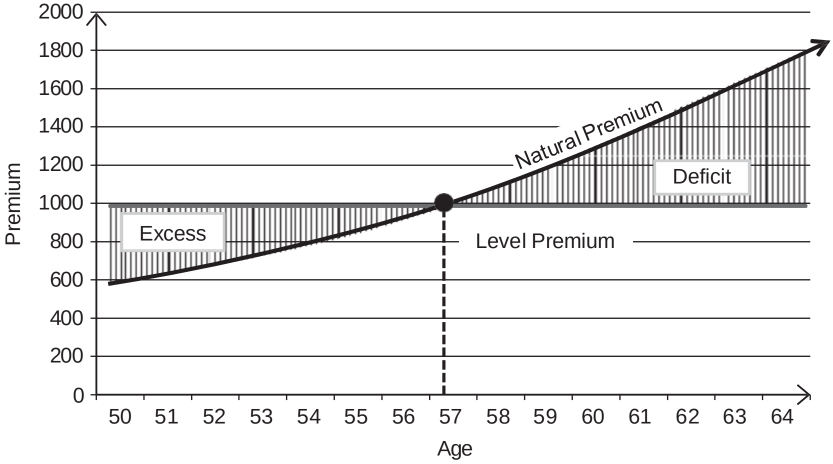

FIGURE 2.1

natural curve up to age 57, then continues below the natural curve until the end of the 15-year term.

4.2.12 RESERVE AND TERMINAL RESERVE FUNDS

The **insurance reserve fund** is the difference between the accumulated value of insurance premiums and the accumulated cost of the insurance. Specifically, the positive excess of the premium in the early years that we saw in Table 2.1 and Figure 2.1 is allowed to accumulate and gain interest so that it can be used to pay the due death claims and other obligations, especially when the level premium accumulations lag behind the liabilities in the later years. Since the amount of reserve, the interest it earns, the death claims, and the obligations are all calculated in each year of the policy term, the remaining fund forms what is called the **terminal reserve**. Therefore, the terminal reserve is what is left in the reserve fund at the end of any policy years after paying off all death claims and other obligations, and after dividing the result by the number of survivors at that time. In this sense, the terminal reserve would represent the maximum amount that a policyholder would expect to get should the discontinuation of the policy be chosen at year's end. It can also serve as the guide to the maximum amount of loan that a policyholder can obtain from his insurance company if he wants to put his policy up as a security asset for such a loan.

There are two major methods of calculating the terminal reserve: the retrospective and the prospective.

The Retrospective Method

Table 2.2 shows in detail how the terminal reserve is obtained in each year of a 15-year term policy of $100,000 issued to a 50-year-old person. This method

TABLE 2.2 Terminal Reserve by the Retrospective Method

(1) Policy Year	(2) Age	(3) l_x	(4) d_x	(5) Total Premium at the Start of the Year [(2) × 986.25]	(6) Reserve at the Start of the Year [(5) + (9) ↑]	(7) Interest Earned on Reserve [(6) × .05]	(8) Death Claims at the End of the Year ($d_x \times 100,000$)	(9) Reserve at End of the Year [(6) + (7) = (8)]	(10) Terminal Reserve [(9) ÷ (3)]
1	50	91,526	540	90,268,432	90,268,432	4,513,422	54,000,000	40,781,854	445.58
2	51	90,986	584	89,735,852	130,268,432	6,525,885	58,400,000	78,643,591	864.35
3	52	90,402	631	89,159,876	167,803,467	8,390,173	63,100,000	113,093,640	1,251.00
4	53	89,771	684	88,537,546	201,631,186	10,081,559	68,400,000	143,312,745	1,596.43
5	54	89,087	739	87,862,944	231,175,689	11,558,784	73,900,000	168,843,473	1,895.16
6	55	88,348	797	87,134,098	255,968,573	12,798,428	79,700,000	189,067,002	2,140.03
7	56	87,551	856	86,348,049	275,415,051	13,770,752	85,600,000	203,585,804	2,325.34
8	57	86,695	919	85,503,810	289,089,614	14,454,481	91,900,000	211,644,095	2,441.25
9	58	85,776	987	84,597,437	296,241,532	14,812,077	98,700,000	212,353,609	2,475.68
10	59	84,789	1,063	83,623,999	295,977,608	14,798,880	106,300,000	204,476,488	2,411.59
11	60	83,726	1,145	82,575,604	287,052,092	14,352,605	114,500,000	186,904,697	2,232.34
12	61	82,581	1,233	81,446,337	268,351,034	13,417,552	123,300,000	158,468,585	1,918.98
13	62	81,348	1,324	80,230,278	238,698,863	11,934,943	132,400,000	118,233,806	1,453.43
14	63	80,024	1,415	78,824,470	197,158,276	9,857,914	141,500,000	65,516,190	820.01
15	64	78,609	1,502	77,528,912	143,045,102	8,582,706	150,200,000	1,427,808	18.16

of obtaining the terminal reserve is called the **retrospective method** since its calculations are made at the end of each year of the policy term by looking backward to the beginning of that year.

As an example of the process to calculate the terminal reserve, let's follow how the terminal reserve of the eighth year of the policy, $2,441.25, was calculated:

3. The total premium at the beginning of the eighth year [column (5)] is calculated by multiplying the level premium of $986.26 by the number of survivors at the beginning of that year (1_x; the level premium was calculated in Section 2.11).

$$986.26 \times 86{,}695 = 85{,}503{,}810$$

4. The figure obtained in step 1 (85,503,810) is added to the figure for the reserve at the end of previous year (the seventh) in column (9), which is 203,585,804:

$$85{,}503{,}810 + 203{,}585{,}804 = 289{,}089{,}614$$

   This would be the entry in column (6).

5. The figure in step 2 would earn 5% interest, which comes to $14,454,481. It is recorded in column (7).

$$289{,}089{,}614 \times .05 = 14{,}454{,}481$$

6. Death claims at the end of the year are obtained by multiplying the face value of the policy, $100,000, by the number of dying (d_x) in column (4).

$$919 \times 100{,}000 = 91{,}900{,}000$$

   This is recorded in column (8).

7. The reserve fund at the end of the eighth year would be obtained by adding the reserve value at the beginning of the year [column (6)], to the interest earned [column (7)], and then subtracting the death claims [column (8)].

$$(289{,}089{,}614 + 14{,}454{,}481) - 91{,}900{,}000 = 211{,}644{,}095$$

   This is now the amount of reserve at the end of the eighth year, recorded in column (9).

8. To get the terminal reserve per survivor, the amount of reserve at the end of the year found in step 5 is divided by the number of people living (1_x) in column (3).

$$211{,}644{,}095 \div 86{,}695 = 2{,}441.25$$

We can also use the following formula to get the terminal reserve for any year (t) without the need to construct a table:

$$\boxed{V = \frac{P(N_x - N_{x+t}) - (M_x - M_{x+t})}{D_{x+t}}}$$

where V is the terminal reserve per survivor at the end of the tth year of the policy term, P is the annual premium per \$1.00, t is any year of the policy term, and the rest are the usual commutation values from Table 10 in the Appendix.

Example 2.12.1 Let's try to check this formula in finding the eighth-year terminal reserve per survivor of Table 2.2.

$x = 50, t = 8, x + t = 58$, and the premium per dollar would be obtained by dividing the annual premium of 986.26 by the face value of the policy, \$100,000.

$$P = \frac{986}{100{,}000} = .00986$$

$$V = \frac{P(N_x - N_{x+t}) - (M_x - M_{x+t})}{D_{x+t}}$$

$$V = \frac{P(N_{50} - N_{58}) - (M_{50} - M_{58})}{D_{58}}$$

$$= \frac{.00986(118{,}106.84 - 65{,}158.16) - (2{,}357.27 - 1{,}959.87)}{5{,}062.95}$$

$$= .0246$$

which is the terminal reserve for \$1.00. So for the face value of \$100,000 it would be

$$.0246 \times 100{,}000 = 2{,}460$$

which is very close to what we obtained in Table 2.2 (2,441). The difference is due to rounding off.

The Prospective Method

Instead of looking backward as we do in the retrospective method, the **prospective method** would look forward to the future benefits and future liabilities of the policy. In this case, the reserve would be the excess of the present value of the future benefits over the present value of the future premiums. This can be translated as the difference between A_{x+t}, which is the value of insurance at age $x + t$, and ${}_tA_x$, which is the value of the annuity due of the annual premiums deferred t years.

$${}_tV_x = A_{x+t} - {}_tA_x$$

$${}_tV_x = F\left(\frac{M_{x+t} - M_{x+n}}{D_{x+t}}\right) - P_x\left(\frac{N_{x+t} - N_{x+n}}{D_{x+t}}\right)$$

$$\boxed{{}_tV_x = \frac{F(M_{x+t} - M_{x+n}) - P_x(N_{x+t} - N_{x+n})}{D_{x+t}}}$$

Example 2.12.2 Find the eighth terminal reserve of the earlier examples using the prospective method.

$$F = 100{,}000; x = 50; t = 8; x + t = 58; n = 15; x + n = 65.$$

$${}_tV_x = \frac{F(M_{x+t} - M_{x+n}) - P_x(N_{x+t} - N_{x+n})}{D_{x+t}}$$

$${}_8V_{50} = \frac{100{,}000\,[M_{58} - M_{65}] - 986.26\,[N_{58} - N_{65}]}{D_{58}}$$

$$= \frac{100{,}000(1{,}959.98 - 1{,}542.78) - 986.26(65{,}158.16 - 35{,}523.27)}{5{,}062.75}$$

$$= 2{,}467$$

which is close to the value we arrived at earlier except for the slight discrepancies caused by rounding off.

4.2.13 BENEFITS OF THE TERMINAL RESERVE

Since the terminal reserve is the value of the insurance policy in any year of the insurance term, it can be serving many financial needs for a policyholder. It can, for example, be used to determine the cash that a policyholder may receive in case of terminating payments or surrendering the policy. It can also be used as a guide to determine the maximum amount of loan that a policyholder may get from her insurance company to take advantage of the interest rate, which is usually much lower than the best rate obtained from a commercial bank. The policy and its reserve would be considered a security asset for the loan. The reserve can also be received and used as a single premium to extend a current policy or get a reduced amount of paid-up insurance.

4.2.14 HOW MUCH LIFE INSURANCE SHOULD YOU BUY?

Life insurance is a protection for survivors who are named as beneficiaries in a policy. It is to provide them with the ability to continue living their lives normally after losing a loved one. There are two approaches to estimating the dollar amount to buy in life insurance.

The Sentimental Approach

The sentimental approach is highly subjective. Basically, it involves the insurance buyer choosing the amount of insurance that reflects the emotional value he places on his survivors — much more than to consider their real needs. In such a case, people may choose $1 million or more just to make a statement that this is the way they want to help their families after the policyholders are dead.

The Rational Approach

In the rational approach, insurance buyers rely on certain criterion as a basis for their consideration to the amount of insurance they buy. They may consider either a very general measure such as the insured's income, or go over more detailed calculations of the needs that survivors may face. Two common methods are used in this rational approach.

The Multiple-Earnings Method This method is simple and straightforward. It depends on the insured's income as a basis of estimation. The person who wants to buy insurance can multiply her income by a certain multiple such as 5 or 10 or any other number so that the final amount would yield the desired life insurance proceeds to the beneficiaries. Table 2.3 shows general estimates of an average premium for each $100,000 in purchased insurance, arranged by several term policies for men and women. For example, if a 50-year-old man wants to buy a 25-year term life insurance equal to five times his annual income of $75,000, the amount of insurance would be:

$$75{,}000 \times 5 = 375.000$$

and his premium would be

$$375{,}000 \div 100{,}000 = 3.75$$

Looking at Table 2.3 across age 50 and term 25 in the men's section, we find the annual premium of $762 per $100,000. Since we have 3.75, the premium would be

$$3.75 \times 762 = \$2{,}857.50 \text{ annually}$$

The Needs Method This method has more objective elements to consider in deciding how much life insurance to buy. It considers the most common factors that would affect the way the survivors will live after the insured's death. It also takes into consideration the time value of money. In addition to the income required for a certain number of years after the insured's death, it considers government benefits, the final expenses related to the insured's death, the re-adjustment expenses for the spouse and children, children's education, family debt (if any), and whatever the insured can specify as an important financial

TABLE 2.3 Average Annual Premium per \$100,000 Term Life Insurance

		Term of Insurance (years)					
Gender	Age	5	10	15	20	25	30
Male	20	144	133	154	166	216	187
	30	152	134	157	172	229	250
	40	195	170	208	235	356	375
	50	253	310	409	467	762	825
	55	510	455	610	750	1,174	1,306
Female	20	125	116	154	166	216	187
	30	135	124	142	148	189	210
	40	176	150	186	204	279	301
	50	272	240	317	358	559	604
	55	343	320	429	514	848	984

element, into the calculation of the amount of insurance needed. The following equation sums up most of these elements; LA is the life insurance amount needed.

$$\boxed{\mathrm{LA} = \mathrm{PV}(.75Y_d) + \mathrm{OT} - [\mathrm{PV}(G) + \mathrm{CI}]}$$

$PV\ (.75Y_d)$: the present value of a certain stream of future income for the survivors. This income is determined by the notion that the survivors would need about 75% of the insured's disposable income (Y_d) in his working years. This assumes that the 25% deducted used to be his share of the family income.

OT: a vector of needed expenses including but not limited to F, R, D, and E, described below.

F : the final expenses related to the funeral and burial of the deceased.

R : the readjustment expenses for spouses and children related to taking time off, training and getting a job, therapy or psychological treatment and the like.

D : the debt expenses to pay off whatever obligations the insured left.

E : the educational expenses related most likely to the college expenses for the insured's children.

$PV\ G$: the present value of the future stream of government benefits, such as Social Security survivor's benefits; this item is to be deducted.

CI: the current insurance policies that exist at the time of the insured's death, such as the employer's insurance and other private insurance (if any); this item is also to be deducted.

The LA equation can be rewritten as

$$\boxed{\text{LA} = \frac{.75Y_d[1-(1+r)^{-n}]}{r} + \text{OT} + \frac{\text{G}[1-(1+r)^{-k}]}{r} + \text{CI}}$$

Y_d : the insured's disposable income.

r : the interest rate at which the discount is calculated. It is often considered the after-tax, after-inflation investment rate.

n : the time estimated for the needs of the survivors. It is determined by the insured, such as the time until the youngest child finishes college, or when the widow turns 65, and the like.

OT: the sum of whatever expenses are outstanding such as $F + R + D + E$.

G : the annual government benefits.

K : the time until government benefits end, which is determined by the Social Security Administration. Unlike n, it is not determined subjectively by the insured.

CI: the existing insurance.

If we use the table values for the present value factor, the LA formula would be

$$\boxed{\text{LA} = (.75Y_d) \cdot a_{\overline{n}|r} + \text{OT} + [G \cdot a_{\overline{k}|r} + \text{CI}}$$

Example 2.14.1 Bryan and Heather are a married couple in their early thirties. They have two young children. Bryan would like his family to live comfortably for at least 25 years after he dies, and he is thinking of buying a life insurance that would make his goal possible. Here are the data elements:

Gross income : \$60,000

Taxes and payroll deductions : \$18,000

Social Security benefits : \$2,400 a month for 14 years

Education fund for children : \$80,000

Debt to be paid off : \$12,000

Readjustment expenses for Heather and children : \$20,000

Final expenses : \$10,000

Given that the couple has existing insurance of \$85,000, how much life insurance would they need if the interest rate is 5%?

$$Y_d = Y_g - (T + D)$$

$$= 60{,}000 - 18{,}000 = 42{,}000$$

$$.75Y_d = .75(42{,}000) = 31{,}500$$

$$n = 25 \qquad r = .05 \qquad k = 14$$

$$G = 2{,}400 \times 12 = 28{,}800$$

$$\text{OT} = E + D + R + F$$

$$\text{OT} = 80{,}000 + 12{,}000 + 20{,}000 + 10{,}000 = 122{,}000$$

$$\text{LA} = \frac{.75Y_d[1 - (1 + r)^{-n}]}{r} + \text{OT} - \frac{G[1 - (1 + r)^{-k}]}{r} + \text{CI}$$

$$= \frac{31{,}500[1 - (1 + .05)^{-25}]}{.05} + 122{,}000$$

$$- \frac{28{,}800[1 - (1 + .05)^{-14}]}{.05} + 85{,}000$$

$$= 443{,}959.25 + 122{,}000 - (285{,}079.7 + 85{,}000)$$

$$= 195{,}879$$

If we use Appendix Table 10 values for the present value:

$$\text{LA} = (.75Y_d) \cdot a_{\overline{n}|r} + \text{OT} - (G \cdot a_{\overline{k}|r} + \text{CI})$$

$$= 31{,}500(a_{\overline{25}|5}) + 122{,}000 - [28{,}800(a_{\overline{14}|5}) + 85{,}000]$$

$$= 31{,}500(14.0939) + 122{,}000 - [28{,}800(9.8986) + 85{,}000]$$

$$= 195{,}878.17$$

UNIT V

Mathematics of Insurance

5.1 Property and Casualty Insurance

Most property and casualty insurance policies cover both property and liability risks. They reimburse the policyholder for particular financial losses that incur due to any damage, destruction, or loss of use of property owned or controlled by the policyholder. Fire, auto accidents, vandalism, and storms are examples of **casualty damage**. Property and casualty insurance also reimburses property damage and bodily injuries maintained by others but for which the policyholder is responsible. We call this **liability insurance**.

When it comes to reimbursement, two major principles guide the reasoning of the insurance concept.

1. The first is that the financial losses have to be a result of pure vs. speculative risk. Pure risk occurs when there is only the potential for loss and no potential for gain, whereas speculative risk involves a potential for both gain and loss, such as what happens in gambling, which cannot be insured for reimbursing the gambler for his losses.
2. The second principle is that the insurance would not pay more than the actual financial loss, which is called the **principle of indemnity**.

The major concern here is on the matter of how to assess the actual losses. Insurance companies have been instituting the rule that they pay what is called the **actual cash value** as opposed to both the original cost of the property to the policyholder and the replacement cost. The actual cash value (ACV) considers depreciation of the property due to wear and tear and market conditions, while the replacement value would consider the original cost and the appreciation value of the property. The actual cash value can be calculated easily by taking the depreciation effect away from the original cost (OC) of the property. The depreciation effect would consider the current age of property (CA) and its life expectancy (LE).

$$\text{ACV} = \text{OC} - \text{CA} \cdot \frac{\text{OC}}{\text{LE}}$$

$$\boxed{\text{ACV} = \text{OC}\left(1 - \frac{\text{CA}}{\text{LE}}\right)}$$

Example 1 A sofa was purchased 4 years ago for $1,500. Determine how much the reimbursement to the policyholder would be, if the insurance adjuster estimates the life expectancy of this sofa by 9 years.

$$\text{OC} = 1{,}500;\ \text{CA} = 4;\ \text{LE} = 9.$$

$$\begin{aligned} \text{ACV} &= \text{OC}\left(1 - \frac{\text{CA}}{\text{LE}}\right) \\ &= 1{,}500\left(1 - \frac{4}{9}\right) \\ &= 833.33 \end{aligned}$$

Example 2 The policyholder in Example 1 was expecting to have his sofa replaced with a new sofa of the type that he lost. Given that the inflation rate has been averaging 1.75% annually, what replacement value would the policyholder hope for?

The replacement value (RV) includes the original cost plus the changes in the original cost (.6.OC) since the sofa was purchased.

$$\boxed{\text{RV} = \text{OC} + .6.\text{OC}}$$

The change in the original cost here is the rise in prices due to inflation, which would be calculated at 1.75% each year since the sofa was purchased 4 years ago.

$$\boxed{.6.\text{OC} + \text{OC} \cdot f \cdot t}$$

where OC is the original cost, f is the annual inflation rate, and t is the time.

$$\begin{aligned} .6.OC &= 1{,}500(.0175)(4) \\ &= 105 \\ \text{RV} &= \text{OC} + .6.\text{OC} \\ &= 1{,}500 + 105 \\ &= 1{,}605 \end{aligned}$$

If we plug .6.OC in to the RV formula, we get

$$\text{RV} = \text{OC} + .6.\text{OC}$$

$$\text{RV} = \text{OC} + \text{OC} \cdot f \cdot t$$

$$\boxed{\text{RV} = \text{OC}(1 + f \cdot t)}$$

which is more convenient than the earlier formula, and we get exactly the same result:

$$\begin{aligned} \mathrm{RV} &= 1{,}500[1 + (.0175)(4)] \\ &= 1{,}605 \end{aligned}$$

Example 3 What would be the replacement value of a television set that was purchased 7 years ago for $1,350? Inflation has averaged $2\frac{1}{4}$ % annually.

$$\begin{aligned} \mathrm{RV} &= \mathrm{OC}(1 + ft) \\ &= 1{,}350[1 + .0225(7)] \\ &= 1{,}562.62 \end{aligned}$$

Again, the insurance companies would not consider the replacement value in their reimbursement but would probably rely on considering the actual cash value.

5.1.1 DEDUCTIBLES AND CO-INSURANCE

A **deductible** is a policyholder's initial share of the insurance compensation and the **co-insurance** is the final policyholder's share of the entire compensation. Whereas the deductible is stated as a lump sum, co-insurance is a percentage share. For example, if a policy states a coverage of 85 : 15, it means that the entire reimbursement is split between the insurer and the insured, where the insurer carries 85% and the insured carries 15%. Both the deductible and co-insurance are also strategies to reduce the final cost to the insurer while promoting and pushing for more responsibility from the insured. People would think twice before filing a claim if they are going to be responsible for part of the cost. Some insurance companies make the deductible optional for a policyholder.

The following is specific to dwelling property insurance, where companies require that full coverage of any losses to a dwelling requires that the property has been insured for at least 80% of its replacement value. The specification of 80% is under the assumption that the land is worth 20% of the property and would not suffer any damage even if the property burns to the ground. This is why the 80% would be considered equal to full insurance for the structure and the contents of the property. Therefore, the reimbursement for the dwelling losses (R_d) would be calculated as:

$$\boxed{R_d = \frac{I}{.80\ \mathrm{RV}}(L - D)}$$

where I is the insurance purchased, L is the total losses on the claim, D is the deductible, and RV is the replacement value of the property.

Example 3.1.1 A fire causes $60,000 in damage to a house whose replacement value is $135,000 but is insured for $108,000 with a $1,000 deductible. Determine the insurance reimbursement.

$I = 108{,}000$; $\mathrm{RV} = 135{,}000$; $L = 60{,}000$; $D = 1{,}000$.

$$\begin{aligned} R_d &= \frac{I}{.80\ \mathrm{RV}}(L - D) \\ &= \frac{108{,}000}{.80(135{,}000)}(60{,}000 - 1{,}000) \\ &= 59{,}000 \end{aligned}$$

The $59,000 is a full insurance coverage in consideration of the deductible. The reason that the insurance fully compensates the loss because the policyholder has purchased full insurance, as 108,000 is exactly 80% of the replacement value.

Let's consider the following example, where the policyholder buys an insurance policy for less than the replacement value.

Example 3.1.2 Suppose that the policyholder in Example 3.1.1 has purchased only $96,000 worth of insurance. How much would the reimbursement be for losses of $60,000?

$$\begin{aligned} R_d &= \frac{I}{.80\ \mathrm{RV}}(L - D) \\ &= \frac{96{,}000}{.80(135{,}000)}(60{,}000 - 1{,}000) \\ &= 52{,}444 \end{aligned}$$

$52,444 would be the reimbursement, commensurate with the amount of coverage the policyholder chose to buy, which is less than the insurance required for full coverage. If we take a closer look, we can quickly discover that because the policyholder purchased only 88.8% of what is required (96,000/108,000 = 88.8%), the insurance made his imbursement as 88.8% of full coverage (52,444/59,000 = 88.8%)

5.1.2 HEALTH CARE INSURANCE

Health care insurance comes in the form of many types of health plans, such as a health maintenance organization (HMO), a preferred provider organization (PPO), a managed care plan (MCP), government health care programs such as Medicare and Medicaid, and private health insurance. Reimbursement to the policyholder is affected first and foremost by the type of plan and the variety of coverage available. In calculating the reimbursement, two elements play a big role: the deductible and the co-insurance. The deductible in the health care insurance domain usually comes in two types: the deductible per event, such as hospitalization or operation, and the annual deductible. The co-insurance is just the percentage share of the insured and can differ from plan to plan. Generally, health insurance reimbursement (R_h) is determined by

$$\boxed{R_h = (I - \mathrm{CP})(L - D)}$$

where CP is the policyholder's co-insurance percentage, L is the total financial loss (entire medical bill), and D is the deductible.

Example 3.2.1 A man had to be hospitalized 6 days for treatment after an accident. The hospital charged \$500 per day, doctors' fees were \$3,620, lab fees were \$987, and ambulance service cost \$340. He has a 80 : 20 insurance plan with a \$500 deductible per event. Determine how much he would expect his insurance to pay and how much would fall on him to pay.

$\mathrm{CP} = 20\%; D = 500.$

First, we have to add up all the medical charges to get L:

$$\begin{aligned} L &= (6 \times 500) + 3{,}620 + 897 + 340 = 7{,}857 \\ R_h &= (1 - \mathrm{CP})(L - D) \\ &= (1 - .20)(7{,}857 - 500) \\ &= 5{,}885.60 \end{aligned}$$

This is the insurance share, which is 80% — not of the entire bill of \$7,857 but of the bill after the deductible, which is \$7,357. This would leave the policyholder with

$$7{,}857 - 5{,}885.60 = 1{,}971.50$$

The policyholder's share of \$1,971.40 is split between \$500 deductible and \$1,471.40 co-insurance. It is important to notice here that the insured pays in reality more than the stated 20% of the bill. In this case, he paid \$1,971.40 out of \$7,857, which is a little more than 25%, and the insurance company paid \$5,885.60 out of \$7,857, which is a little less than 75% of the bill rather than the stated share of 80%. The reason for this discrepancy, of course, is the deductibles! The 20% share of the policyholder is the \$1471.40 out of the total bill after the deductibles [1471.40/(7857−500)].

The policyholder's copayment may come in different categories and different designations. For example, the copayment for a family practitioner's visit is different from the copayment for a specialist's visit. Also, there are different copayments for different medications and different services, such as lab fees and the like. In an aggregate sense, the insurance reimbursement would be to carry

TABLE E3.2.2

Category	n	k	nk	CP	nCP	k – CP	$n(k$ – CP)
Family practitioner	8	75	600	15	120	60	480
Specialist	2	105	210	35	70	70	140
X-ray	4	59	236	20	80	39	156
Lab report	3	70	210	30	90	40	120
Prescription	12	60	720	15	180	45	540
Dietitian	2	55	110	10	20	45	90
			2,086		560	299	1,526

the insurer's share after the policyholder pays his share of the stated co-insurance of a variety of services, plus the deductibles.

Let's consider the following example, which illustrates how to calculate the insurance and the policyholder's shares of a variety of medical services, each of which has its own policy-allowed co-insurance.

Example 3.2.2 Suppose that a patient incurred the following medical expenses in eight visits to his family practitioner at $75 each: two visits to specialists at $105 each, four x-rays at $59 each, three lab reports at $70 each, 12 prescriptions at an average of $60, and two visits to a dietitian at $55 each. The insurance policy specifies the copayments for each category as the following: $15 for a family practitioner; $35 for a specialist, $20 for an x-ray, $30 for a lab service, $15 for a prescription, and $10 for a professional service. Calculate the insurance reimbursement, the policyholder share, and the percentages of both.

The best way to calculate this was to organize the information in Table E3.2.2, where:

n: the number of unit for each category.
k: the cost per unit.
nk: the cost per category.
CP: the co-insurance per category as specified in the policy.
nCP: the co-insurance cost per category.
k – CP: the difference between the cost and the co-insurance per unit.
$n(k$ – CP$)$: the difference between the cost and the co-insurance per category.

This is, in fact, what the insurance company pays per category because they carry the cost after the policyholder's share has been paid. In total, the cost of all categories $(\sum nk)$ is $2,086 and the total co-insurance cost $(\sum n\cdot \text{CP})$ is $560, which is the policyholder's share, and the difference would be the insurance share or the reimbursement (R_n):

$$R_n = \sum (nk) - \sum (n \cdot \mathrm{CP})$$
$$= 2{,}086 - 560$$
$$= 1{,}526$$

The reimbursement equation above can be rewritten as

$$R_n = \sum nk - \sum n\mathrm{CP}\ R_n$$

$$\boxed{= \sum n(k - \mathrm{CP})}$$

which is the total of the last column $n(k - \mathrm{CP})$, \$1,526, to confirm the result. As for the percentages, the insurance company would pay 73% $(1{,}526 \div 2{,}086)$, and the policyholder would pay the rest, 27% $(560 \div 2{,}086)$.

5.1.3 POLICY LIMIT

Most insurance policies specify a certain limit to what can be reimbursed. The limit is either imposed per event or per year or both. Let's suppose that the insurance in Example 3.2.2 has specified a policy limit of \$1,350 per event (case of illness or injury and the like). Under this policy, the insurance would not pay its share of \$1,526, as calculated. It would pay up to the maximum limit, which is, as specified, only \$1,350. The policyholder has to carry the difference of \$176 (\$1,526 – \$1,350).

The annual limit can work in the same manner. Suppose that there is an annual policy limit of \$12,000 and that throughout the year the insurance has paid \$11,000 for many other cases for this patient. Now that this patient has a new case of treatment in which the insurer's share comes to \$1,526. The insurance would stick to what is left from their coverage limit of \$12,000 which is only \$1,000. They would only pay \$1,000 out of their calculated share of \$1,526. The policyholder has to carry the rest, \$526. In these two examples of imposing the policy limit, the policyholder would end up paying a higher percentage of co-insurance than is stated in the policy.

Summary

This final unit dealt with the mathematics of insurance. We started with life annuities, which are distinct from the earlier annuities certain by being contingent and related to life insurance. We discussed a mortality table, which is a central concept for understanding and calculating all life annuities and insurance. This was followed by an explanation of the commutation terms, which are other important complements to the mortality table, conceptually and technically. Pure endowment as a single payment preceded the discussion of all the common types of life annuities. We discussed and calculated the whole life annuities: ordinary, due, and deferred. The temporary life annuities have the same subdivision: ordinary, due, and deferred.

Life insurance was the subject of the second chapter in the unit. Three major types of life insurance were discussed: the whole life policy, the term policy, and the endowment policy. We calculated the annual premium for regular payments and for m payments, which had the same pattern of deferred premium. The term life policy was discussed next, followed by the endowment policy. A nonannual premium was also discussed and the natural vs. level premium comparison was made. The concepts of reserve and terminal reserve came next in the discussion and their calculation included two methods: retrospective and prospective.

The last topic in the life insurance chapter was how to estimate what is needed to purchase life insurance. Here, four approaches were explained: the sentimental, rational, multiple earnings, and needs approaches.

The last types of insurance discussed in this unit were property and casualty and health care insurance. Two important technical terms were discussed and calculated: actual cash value and replacement value, as well as deductibles, co-insurance, and policy limits.